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PROJECTIVE PLANES AND THEIR LOOK-ALIKES

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Abstract

We classify all closed topological manifolds which have the same integral homology as a projective plane.

In this paper we classify manifolds which look like projective planes. More precisely, we consider 1-connected closed topological manifolds M with integral homology

$$H_\bullet(M) \cong \mathbb{Z}^3.$$

A straight-forward application of Poincaré duality shows that for such a manifold there exists a number $m \geq 2$ such that $H_k(M) = \mathbb{Z}$, for $k = 0, m, 2m$; in particular, $\dim(M) = 2m$ is even. It follows from Adams' Theorem on the Hopf invariant that m divides 8.

We construct a family of topological $2m$-manifolds $M(\xi)$ which are Thom spaces of certain topological $\mathbb{R}^m$-bundles (open disk bundles) ξ over the sphere $\mathbb{S}^m$, for $m = 2, 4, 8$, and which we call models. This idea seems to go back to Thom and was exploited further by Shimada [53] and Eells-Kuiper [14]. A particular case is worked out in some detail in Milnor-Stasheff [46, Ch. 20]. However, these authors used vector bundles instead of $\mathbb{R}^m$-bundles. We will see that the non-linearity of $\mathbb{R}^m$-bundles yields many more manifolds than the construction by Eells-Kuiper. In [14, p. 182], the authors expressed the hope that *"the given combinatorial examples form a complete set [...] for $n \neq 4$"*. Our results show that in dimension $n = 2m = 16$, their construction missed 27/28 of the (infinitely many) combinatorial and topological solutions, while

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in dimension $n = 8$, they obtained all combinatorial, but only half of the topological solutions.

Next, we determine certain characteristic classes of our models, and in particular their rational Pontrjagin classes. Using these characteristic classes and Wall's surgery sequence, we show that for $m \neq 2$, every manifold which looks like a projective plane is homeomorphic to one of our models. Thus, we obtain a complete homeomorphism classification of all manifolds which are like projective planes. Furthermore, we determine which models admit DIFF (or PL) structures. The case $m = 2$ (so $\dim(M) = 4$) is different, but there, we can apply Freedman's classification of closed 4-manifolds.

We also classify the homotopy types of 1-connected Poincaré duality complexes X with $H_\bullet(X) \cong \mathbb{Z}^3$. This homotopy-theoretic version of our main result (which was already proved in Eells-Kuiper [14]) is needed in the course of the homeomorphism classification; since the methods here are somewhat different from the rest of the paper, I put it in an appendix.

Main results.

Let M be a 1-connected closed topological manifold which looks like a projective plane, i.e., $H_\bullet(M) \cong \mathbb{Z}^3$. Then $\dim(M) = 2m = 4, 8, 16$.

If $m = 2$, then M is homeomorphic to the complex projective plane $\mathbb{C}\mathrm{P}^2$ or to the Chern manifold Ch^4. These two manifolds are topologically distinguished by their Kirby-Siebenmann numbers $\mathrm{ks}[M] \in \mathbb{Z}/2$; the Chern manifold $(\mathrm{ks}[\mathrm{Ch}^4] \neq 0)$ admits no DIFF structure (this case is due to Freedman [19]).

If $m = 4$, then M is homeomorphic to one of our models $M(\xi)$. Topologically, it is determined by the Pontrjagin number $p_4^2[M] \in \{2(1+2t)^2 \mid t \in \mathbb{Z}\}$ and the Kirby-Siebenmann number $\mathrm{ks}^2[M] \in \mathbb{Z}/2$. These data also determine the oriented bordism class of M, so no two models are equivalent under oriented bordism. The manifold admits a PL structure (unique up to isotopy) if and only if $\mathrm{ks}^2[M] = 0$.

If $m = 8$, then M is homeomorphic to one of our models $M(\xi)$. Topologically, it is determined by the Pontrjagin number

$$p_8^2[M] \in \left\{ \frac{36}{49}(1+2t)^2 \mid t \in \mathbb{Z} \right\}$$

and a characteristic number $\frac{7}{6}p_8\kappa[M] \in \mathbb{Z}/4$, determined by the integral characteristic class $\frac{7}{6}p_8(M)$ and a certain 8-dimensional PL characteristic class κ with $\mathbb{Z}/4$-coefficients. These data also determine the oriented *bordism class of M, so no two models are equivalent under oriented bordism. These manifolds admit a* PL *structure* (*unique up to isotopy*).

We determine also which of these manifolds admit a DIFF structure, and determine the homotopy type in terms of the characteristic classes. See Sections 7 and 8 for more detailed statements. As a by-product of our proof, we obtain an explicit classification of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$ in terms of characteristic classes, for $m = 2, 4, 8$.

* * *

Topological geometry plays no rôle in this paper. However, the motivation to write it came from a long-standing open problem in topological geometry:

($*$) *What are the possible homeomorphism types of the point spaces of compact projective planes* (*in the sense of Salzmann* [52])?

The point space P of a compact projective plane is always the Thom space of a locally compact fiber bundle, see Salzmann *et al.* [52] Ch. 5, in particular 51.23. (Problem ($*$) should not be confused with the *geometric* problem of classifying all compact projective planes with large automorphism groups which was solved by Salzmann and his school [52].)

Now ($*$) turns out to be a difficult problem. The present state of affairs is as follows, see [52]. Let P be the point space of a compact projective plane. If the covering dimension of P is $\dim(P) = 0$, then P is either finite or homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$. If $1 \leq \dim(P) \leq 4$, then $P \cong \mathbb{R}\mathrm{P}^2$ or $P \cong \mathbb{C}\mathrm{P}^2$; this was proved by Salzmann and Breitsprecher already in the late 60s [7] (surprisingly, this did not require results about 4-manifolds). The proof depends on a result by Borsuk about low-dimensional ANRs and on Kneser's Theorem $\mathrm{SO}(2) \simeq \mathrm{STOP}(2)$. In (finite) dimensions bigger than 4, Löwen [40] applied sheaf-theoretic cohomology to the problem. Using a beautiful local-to-global argument, he proved that P is an $m-1$-connected Poincaré duality complex and an integral $2m$-dimensional ENR manifold with $H_\bullet(P) \cong \mathbb{Z}^3$, and that $m = 2, 4, 8$.

So the topological problem ($*$) is reduced to the following steps:

(1) Prove that the topological dimension $\dim(P)$ is finite.

(2) Assuming that $\dim(P) < \infty$, prove that P is a manifold (and not just an integral ENR manifold).

(3) Assuming that P is a manifold, determine its homeomorphism type.

Each step seems to be difficult. Under the additional assumption that the compact projective plane is *smooth* (in the sense of [52]: the geometric operations are smooth maps), a complete homeomorphism classification (based on characteristic classes) of the point spaces was carried out in [36]. Buchanan [9] determined the homeomorphism types of the point spaces of compact projective planes coordinatized by real division algebras by a direct homotopy-theoretic argument (note that besides the classical alternative division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, there exists a continuum of other real division algebras). In both cases, the homeomorphism types of the point spaces turn out to be the classical ones, $\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{O}\mathrm{P}^2$. My hope is that the results in this paper, together with Knarr's Embedding Theorem [35] and the result in [37] will eventually lead to a solution of (3).

* * *

I have tried to make the paper self-contained and accessible to non-experts. There is necessarily a certain overlap with the paper by Eells-Kuiper [14]. My aim was to give complete proofs for all steps of the classification, starting only from general facts about bundles and manifolds. Thus, the reader is not assumed to be familiar with [14] (although this fundamental paper is certainly to be recommended).

Standing assumptions. An n-manifold (without boundary) is a metrizable, second countable space which is locally homeomorphic to $\mathbb{R}^n$. Throughout, all maps are assumed to be continuous. Except for the appendix, maps and homotopies are not required to preserve base points, unless the contrary is stated explicitly. Thus $[X;Y]$ denotes the set of all free homotopy classes of maps from X to Y. If X, Y are well-pointed spaces, and if Y is 0-connected, then the fundamental group $\pi_1(Y)$ acts on the set $[X;Y]_0$ of based homotopy classes; the set $[X;Y]$ of all free homotopy classes can be identified with the orbit set of this action, see Whitehead [66] Ch. III.1. If Y is an H-space (or if Y is 1-connected) this action is trivial, so $[X;Y] = [X;Y]_0$.

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1. Preliminaries on bundles

Fiber bundles, microbundles, fibrations, and Thom spaces play a prominent rôle in this paper, so we briefly recall the relevant notions. We refer to Holm [26], Milnor [43], Dold [11], and to the books by Kirby-Siebenmann [33], Rudyak [50], and Husemoller [30].

1.1. A *bundle* $\phi = (E, B, p)$ over a space B is a map $E \xrightarrow{p} B$. The class of all bundles over B forms in an obvious way a category whose morphisms $\phi \xrightarrow{f} \phi'$ are commutative diagrams

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

An isomorphism in this category is called an *equivalence* of bundles and denoted $\phi \cong \phi'$; in the diagram above, f is an equivalence if and only if f is a homeomorphism. The categorical product of two bundles ϕ, ϕ' is the *Whitney sum* $\phi \oplus \phi'$; its total space is $E \oplus E' = \{(e, e') \in E \times E' \mid p(e) = p'(e')\}$, with the obvious bundle projection. A homotopy between two morphisms $f_0, f_1 : \phi' \rightrightarrows \phi$ or *homotopy over* B is a homotopy $E' \times [0, 1] \longrightarrow E$ with the property that the diagram

$$\begin{tikzcd} E' \arrow[r, "f_t"] \arrow[d] & E \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

commutes for all $t \in [0, 1]$. A morphism f is called a *fiber homotopy equivalence* if it has a homotopy inverse bundle map g, i.e., if fg and gf are homotopic over B to the respective identity maps; in this case we write $\phi \simeq \phi'$. A *section* of a bundle $\phi = (E, B, p)$ is a morphism s from the identity bundle (B, B, id_B) to ϕ,

$$\begin{tikzcd} & E \arrow[d, "p"] \\ B \arrow[ur, "s"] \arrow[r, equal] & B. \end{tikzcd}$$

and we call (E, B, p, s) a *sectioned bundle*.

A map $B' \xrightarrow{g} B$ induces a contravariant functor g^* which assigns to every bundle $\phi = (E, B, p)$ the *pull-back bundle* $g^*\phi = (g^*E, B', p')$, with $g^*E = \{(e, b') \in E \times B' \mid p(e) = g(b')\}$ and $p'(e, b') = b'$. If g is a homeomorphism, then g^* is an equivalence of categories; in this case, two bundles ϕ and ϕ' are called *weakly equivalent* if ϕ' is equivalent to $g^*\phi$, in other words, if there are homeomorphisms

$$\begin{tikzcd} E' \arrow[r, "f", "\cong"'] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "g", "\cong"'] & B \end{tikzcd}$$

commuting with the bundle projections; such a weak equivalence is denoted $\phi \cong_g \phi'$.

1.2. A bundle is called a *fibration* if the homotopy extension problem

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "g"] \arrow[ur, dotted] & B \end{tikzcd}$$

has a solution for every space X. We call a fibration n*-spherical* if every fiber $E_b = p^{-1}(b)$ has the homotopy type of an n-sphere.

For a subspace $A \subseteq B$, we have the *restriction* $\phi|_A = (E_A = p^{-1}(A), A, p|_{E_A})$ of the bundle ϕ.

Definition 1.3. A bundle is called a *fiber bundle* with typical fiber F if every $b \in B$ has an open neighborhood U such that the restriction $\phi|_U$ is equivalent to the product (or trivial) bundle $(F \times U, U, \mathrm{pr}_2)$

$$\begin{tikzcd} F \times U \arrow[r, "\cong"'] \arrow[d, "pr_2"'] & E_U \arrow[r, hook] \arrow[d] & E \arrow[d] \\ U \arrow[r, equal] & U \arrow[r, hook] & B; \end{tikzcd}$$

such a local trivialization is also called a *coordinate chart* for the bundle. If in addition a base point is fixed in the fiber F, one obtains in an obvious way a *sectioned fiber bundle.*

For technical reasons, it is often convenient to consider numerable fiber bundles. For example, every numerable fiber bundle is automatically a fibration, see Spanier [55] Ch. 2.7 Theorem 12.

Definition 1.4. A locally finite covering $\{V_i \mid i \in I\}$ of B by open sets is called *numerable* if there exist maps $f_i : B \longrightarrow [0,1]$ with $f_i^{-1}((0,1]) = V_i$, such that $\sum_{i\in I} f_i = 1$. A fiber bundle is called *numerable* if there exists a numerable covering of B by coordinate charts. In our setting, most base spaces will be paracompact, so fiber bundles are automatically numerable.

Definition 1.5. An *n-sphere bundle* is a numerable fiber bundle with $\mathbb{S}^n$ as typical fiber. An *$\mathbb{R}^n$-bundle* is a sectioned numerable bundle with $(\mathbb{R}^n, 0)$ as typical fiber; the section is denoted s_0 and called the *zero-section.* The trivial $\mathbb{R}^n$-bundle (over any space) will be denoted $\underline{\mathbb{R}}^n$; its total space is $E = \mathbb{R}^n \times B$, with $p = \mathrm{pr}_2$. An n-dimensional *vector bundle* is an $\mathbb{R}^n$-bundle which carries in addition a real vector space structure on each fiber which is compatible with the given coordinate charts. Two $\mathbb{R}^n$-bundles (or vector bundles) ξ, ξ' are called *stably equivalent* if there is an equivalence $\xi \oplus \underline{\mathbb{R}}^k \cong \xi' \oplus \underline{\mathbb{R}}^{k'}$, for some $k, k' \geq 0$.

A crucial property of $\mathbb{R}^n$-bundles is the following homotopy property.

Lemma 1.6. *Let ξ be an $\mathbb{R}^n$ bundle over B. If $g_0, g_1 : B' \rightrightarrows B$ are homotopic, then there is an equivalence $g_0^*\xi \cong g_1^*\xi$.*

Proof. See Holm [26] Lemma 1.5. q.e.d.

1.7. In an $\mathbb{R}^n$-bundle $\xi = (E, B, ps_0)$, the zero-section s_0 is a homotopy inverse to the bundle projection p, and $s(B)$ is a strong deformation retract of the total space E, see Holm [26] Theorem 3.6. In particular, the section $s_0 : B \longrightarrow E$ is a cofibration. It follows that the quotient $E/s_0(B)$ is contractible.

Definition 1.8. From each $\mathbb{R}^n$-bundle ξ, one obtains an n-sphere bundle ${}^s\xi$ by compactifying each fiber of ξ. The resulting bundle ${}^s\xi$ has two sections, the zero-section s_0 and the section s_∞ corresponding to the new points added in the fibers. Let E denote the total space of ξ, and put $E_0 = E \setminus s_0(B)$, the total space with the zero-section removed. Finally, let ${}^uE = E_0 \cup s_\infty(B)$. Then clearly, ${}^u\xi = ({}^uE, B, {}^u p, s_\infty)$ is again an $\mathbb{R}^n$-bundle (called the *upside down bundle* in [37]), and $E_0 \longrightarrow B$ is a numerable fiber bundle with $\mathbb{R}^n \setminus 0$ as typical fiber. We call $E_0 \longrightarrow B$ the *spherical fibration* corresponding to the $\mathbb{R}^n$-bundle ξ. The *Thom* *space* $M(\xi)$ of an $\mathbb{R}^n$-bundle ξ is the quotient

$$M(\xi) = E \cup s_\infty(B)/s_\infty(B) = E \cup \{o\}.$$

We denote the base point (the tip) of $M(\xi)$ by o. If B is compact, then $M(\xi) = E \cup \{o\}$ is the same as the one-point compactification of E.

1.9. By the previous remarks, o is a strong deformation retract of $E_0 \cup \{o\}$. In particular, there is a natural (excision) isomorphism $H^\bullet(E, E_0) \cong H^\bullet(M(\xi), o)$.

We need also the concept of a microbundle, see Milnor [43] and Holm [26].

Definition 1.10. An *n-microbundle* $\mathfrak{x} = (E, B, p, s)$ is a sectioned bundle, subject to the following condition: for every $b \in B$ there exists an open neighborhood U of b, an open subset $V \subseteq E_U = p^{-1}(U)$ containing $s(U)$ and a section-preserving homeomorphism $h : U \times \mathbb{R}^n \longrightarrow V$ such that the diagram

$$\begin{tikzcd}
\mathbb{R}^n \times U \arrow[r, "h"] \arrow[d, "pr_2"'] & V \arrow[r, hook] \arrow[d] & E_U \arrow[r, hook] \arrow[d] & E \arrow[d] \\
U \arrow[r, equal] & U \arrow[r, equal] & U \arrow[r, hook] & B;
\end{tikzcd}$$

commutes. The difference between a microbundle and an $\mathbb{R}^n$-bundle is that h need not be surjective onto E_U. Similarly as for $\mathbb{R}^n$-bundles, we require the existence of a numerable covering of B by such local charts.

Clearly, every $\mathbb{R}^n$-bundle is an n-microbundle. It is also clear that there exist microbundles which are not fiber bundles. A particularly important example is the *tangent microbundle* $\mathfrak{t}M$ of a manifold M: here, $E = M \times M$, the bundle projection is pr_2 and the section is the diagonal map, $s(x) = (x, x)$, see Milnor [43] Lemma 2.1.

The Kister-Mazur Theorem (see Theorem 1.11 below) says that microbundles are in a sense equivalent to $\mathbb{R}^n$-bundles, a fact which is not obvious at all. If $E' \subseteq E$ is a neighborhood of $s(B)$, then it is not difficult to see that $E' \longrightarrow B$ is again a microbundle $\mathfrak{x}'$ *contained in* $\mathfrak{x}$. Two microbundles $\mathfrak{x}_1, \mathfrak{x}_2$ over the same base B are called *micro-equivalent* if they contain microbundles $\mathfrak{x}'_1, \mathfrak{x}'_2$ which are equivalent as bundles (this is also sometimes called a micro-isomorphism or an isomorphism germ).

In the case of numerable microbundles one has to be careful: a microbundle contained in a numerable microbundle need *a priori* not be numerable.

Theorem 1.11 (Kister-Mazur). *Let $\mathfrak{x}$ be a numerable n-microbundle. Then there exists a numerable microbundle $\mathfrak{x}'$ contained in $\mathfrak{x}$ which is an $\mathbb{R}^n$-bundle, and $\mathfrak{x}'$ is unique up to equivalence.*

Proof. See Holm [26] Theorem 3.3. q.e.d.

In particular, the tangent microbundle $\mathfrak{t}M$ of any (metrizable) n-manifold M contains an $\mathbb{R}^n$-bundle, unique up to equivalence. We will call this $\mathbb{R}^n$-bundle τM (and any bundle equivalent to it) the *tangent bundle* of M. If M happens to be a smooth manifold, one can show that τM is equivalent to the smooth tangent bundle TM, see Milnor [43] Theorem 2.2.

If ξ is an $\mathbb{R}^n$-bundle over a manifold B, then the total space E is clearly a manifold, and the zero-section $s_0(B)$ is a submanifold with normal (micro) bundle ξ, see Milnor [43] Sec. 5. We require the following splitting result.

Proposition 1.12. *There is an equivalence*

$$s_0^*\tau E \cong \tau B \oplus \xi.$$

Proof. This follows from Milnor [43] Theorem 5.9, combined with the Kister-Mazur Theorem 1.11 above. q.e.d.

2. Constructing the models

In this section we construct a family of manifolds as Thom spaces of $\mathbb{R}^m$-bundles over the sphere $\mathbb{S}^m$, for $m = 2, 4, 8$, which we call *models*. We begin with some general remarks about Thom spaces of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$. We fix a generator $[\mathbb{S}^m] \in H_m(\mathbb{S}^m)$. Let ξ be an $\mathbb{R}^m$-bundle over $\mathbb{S}^m$, for $m \geq 2$, with total space E, and let $E_0 = E \setminus s_0(\mathbb{S}^m)$ denote the total space with the zero-section removed. Since $\mathbb{S}^m$ is 1-connected, the bundle ξ is orientable, and we may choose an *orientation class* $u(\xi) \in H^m(E, E_0)$, see Spanier [55] Ch. 5.7 Corollary 20. The image $e(\xi) = s_0^\bullet(u(\xi)|_E)$ of $u(\xi)$ in $H^m(\mathbb{S}^m)$ is the *Euler class* of ξ. We call the integer

$$|e| = |\langle e(\xi), [\mathbb{S}^m]\rangle|$$

the *absolute Euler number* of ξ; it is independent of the choice of $[\mathbb{S}^m]$ and of $u(\xi)$. Let $M(\xi)$ denote the Thom space of ξ, and let

$$\Phi : H^\bullet(B) \xrightarrow{\cong} H^{\bullet+m}(E, E_0), \qquad \Phi(v) = p^\bullet(v)\smile u(\xi)$$

denote the Gysin-Thom isomorphism, see Spanier [55] Ch. 5.7 Theorem 10. By 1.9, this yields — via excision — isomorphisms

$$H^\bullet(\mathbb{S}^m) \cong H^{\bullet+m}(E, E_0) \cong H^{\bullet+m}(M(\xi), o).$$

Let y_m, y_{2m} be generators for the infinite cyclic groups $H^m(M(\xi))$ and $H^{2m}(M(\xi))$, respectively.

Lemma 2.1. *In the cohomology ring $H^\bullet(M(\xi))$, we have the relation $y_m^2 = \pm|e|y_{2m}$.*

Proof. Let $x \in H^m(\mathbb{S}^m)$ be the generator dual to $[\mathbb{S}^m]$ and let Φ denote the Gysin-Thom isomorphism. Thus $\Phi(1) = u(\xi) = \pm y_m$ and $\Phi(x) = \pm y_{2m}$. Let $e(\xi) = \varepsilon x$, for $\varepsilon \in \mathbb{Z}$ (so $|e| = |\varepsilon|$). Then $\Phi(e(\xi)) = \varepsilon\Phi(x) = u(\xi)\smile u(\xi)$, since $p^\bullet(e(\xi)) = u(\xi)|_E$. q.e.d.

2.2. If m is odd, then $u(\xi) \smile u(\xi) = 0$, so $e(\xi) = 0$. Therefore, $|e| = 0$ if m is odd.

Proposition 2.3. *If $M(\xi)$ is a manifold, then $|e| = 1$ and m is even and divides 8. Moreover, $H^\bullet(M(\xi); R) \cong R[y_m]/(y_m^3)$ for any commutative ring R.*

Proof. If $M(\xi)$ is a manifold with fundamental class μ, then Poincaré duality implies that the map

$$H^m(M(\xi)) \otimes H^m(M(\xi)) \longrightarrow \mathbb{Z}, \qquad u \otimes v \longmapsto \langle u\smile v, \mu\rangle$$

is a duality pairing, so $|e| = 1$ and m is even. Thus $H^\bullet(M(\xi); R) \cong R[y_m]/(y_m^3)$ for any commutative ring R. Since $M(\xi)$ is a manifold, it is an ANR, see Hanner [21] Theorem 3.3 or Hu [28] p. 98 and thus homotopy equivalent to a CW-complex X, see Weber [65] p. 218; by standard obstruction theory, $X \simeq \mathbb{S}^m \cup e^{2m}$ is homotopy equivalent to a 2-cell complex, see Wall [63] Proposition 4.1. By Adams-Atiyah [2] Theorem A, this implies that $m = 2, 4, 8$. More details can be found in the appendix. q.e.d.

The exact homotopy sequence of the $m-1$-spherical fibration $E_0 \longrightarrow \mathbb{S}^m$ shows that $\pi_1(E_0)$ is abelian, and that E_0 is $m-2$-connected.

For $|e| = 1$, the Gysin sequence

$$\longrightarrow H_k(E_0) \longrightarrow H_k(\mathbb{S}^m) \xrightarrow{e(\xi)\frown} H_{k-m}(\mathbb{S}^m) \longrightarrow H_{k-1}(E_0) \longrightarrow$$

breaks down into isomorphisms $H_0(E_0) \cong H_0(\mathbb{S}^m)$ and $H_m(\mathbb{S}^m) \cong H_{2m-1}(E_0)$, and all other homology groups of E_0 vanish. A repeated application of the Hurewicz isomorphism shows that E_0 is $2(m-1)$-connected, with $\pi_{2m-1}(E_0) \cong H_{2m-1}(E_0) \cong \mathbb{Z}$. Thus

$$H_k(E_0 \cup \{o\}, E_0) \cong \widetilde{H}_{k-1}(E_0) \cong \begin{cases} \mathbb{Z} & \text{for } k = 2m \\ 0 & \text{else,} \end{cases}$$

because $E_0 \cup \{o\}$ is contractible. In other words, $M(\xi)$ has for $|e| = 1$ the same local homology groups at o as $\mathbb{R}^{2m}$.

Recall that, due to the embeddability of second countable finite dimensional metric spaces, a locally compact finite dimensional second countable ANR (absolute neighborhood retract for the class of metric spaces see Hu [28]) is exactly the same as an ENR (Euclidean neighborhood retract, see Hurewicz-Wallman [29] Ch. V, Engelking [17] Theorem 1.11.4, and Dold [13] Ch. IV.8).

Lemma 2.4. *For $|e| = 1$, the Thom space $M(\xi) = E \cup \{o\}$ is an integral* ENR *$2m$-manifold, i.e., an* ENR (*Euclidean neighborhood retract*) *which has the same local homology groups as $\mathbb{R}^{2m}$.*

Proof. The space $E \cup s_\infty(\mathbb{S}^m)$ is a $2m$-manifold (and in particular an ENR), and $M(\xi) = E \cup \{o\} = E \cup s_\infty(\mathbb{S}^m)/s_\infty(\mathbb{S}^m)$ is a quotient of an ENR (the manifold $E \cup s_\infty(\mathbb{S}^m)$) by a compact ENR subspace (the m-sphere $s_\infty(\mathbb{S}^m)$). Such a quotient is again an ENR, see Hanner [21] Theorem 8.2 or Hu [28] Ch. IV. The local homology groups at o were determined above; every point in E has a locally Euclidean neighborhood and thus the same local homology groups as $\mathbb{R}^{2m}$. q.e.d.

Our next aim is to show that $M(\xi)$ is in fact a manifold. Since $M(\xi) = E \cup \{o\}$ and E is a manifold, the only point which we have to consider in detail is o. First, we prove that o is $2(m-1)$-LC in $M(\xi)$, i.e., that every open neighborhood V of o contains an open neighborhood V' of o such that for $k \le 2(m-1)$, every map $\mathbb{S}^k \longrightarrow V' \setminus \{o\}$ is homotopic in $V \setminus \{o\}$ to a constant map. Clearly, we are done if we can show that $V' \setminus \{o\}$ is $2(m-1)$-connected.

Lemma 2.5. *The space $M(\xi)$ is $2(m-1)$-*LC *at o if $|e| = 1$.*

Proof. Let ${}^uE = E_0 \cup s_\infty(\mathbb{S}^m)$ denote the upside-down bundle obtained from ξ (see Definition 1.8). Then ${}^uE \longrightarrow \mathbb{S}^m$ is an $\mathbb{R}^m$ bundle and in particular a microbundle (see Definition 1.10). Let V be an open neighborhood of o in $M(\xi)$, and let f denote the map ${}^uE \longrightarrow E_0 \cup \{o\} \subseteq M(\xi)$ which collapses the s_∞-section to the point o. Then $U = f^{-1}(V)$ is an open neighborhood of $s_\infty(\mathbb{S}^m)$ in the upside-down bundle ${}^uE$. By the Kister-Mazur Theorem 1.11, there exists an open neighborhood U' of $s_\infty(\mathbb{S}^m)$ in U with ${}^uE \supseteq U \supseteq U'$, such that $U' \longrightarrow \mathbb{S}^m$ is an $\mathbb{R}^m$-bundle equivalent to ${}^uE \longrightarrow \mathbb{S}^m$. In particular, $U' \setminus s_\infty(\mathbb{S}^m) \cong E_0$ is $2(m-1)$-connected. Now we put $V' = f(U')$. q.e.d.

Corollary 2.6. *For $|e| = 1$, the Thom space $M(\xi)$ is a 1-connected closed 2m-manifold.*

Proof. The space $M(\xi) \setminus \{o\}$ is a $2m$-manifold, and $M(\xi)$ is 1-LC at o. Thus, o has an open neighborhood homeomorphic to $\mathbb{R}^{2m}$; for $m = 2$, this follows from Freedman-Quinn [20] Theorem 9.3A (and also from Kneser's Theorem $\mathrm{TOP}(2) \simeq \mathrm{O}(2)$, see Theorem 6.4 below), and for $m = 4, 8$ from Quinn [49] Theorem 3.4.1. Van Kampen's Theorem, applied to the diagram

$$\begin{tikzcd} & E_0 \arrow[dl] \arrow[dr] & \\ E \arrow[dr] & & E_0 \cup \{o\} \arrow[dl] \\ & M(\xi) & \end{tikzcd}$$

shows that $\pi_1(M(\xi)) = 0$, because E, $E_0 \cup \{o\}$, and E_0 are 1-connected. q.e.d.

Definition 2.7. The manifolds $M(\xi)$ obtained in this way as Thom spaces of $\mathbb{R}^m$-bundles with $|e| = 1$ will be called *models*.

The same argument as above shows that $E/s_0(\mathbb{S}^m)$ is a manifold, and so $S = M(\xi)/s_0(\mathbb{S}^m)$ is a manifold, too. Similarly as above, Van Kampen's Theorem shows that S is 1-connected. As E_0 has the same homology as $\mathbb{S}^{2m-1}$, the exact homology sequence of the pair (E, E_0) shows that the composite $\mathbb{S}^m \xrightarrow{s_0} E \longrightarrow (E, E_0)$ is an isomorphism in m-dimensional homology. Thus $\widetilde{H}_\bullet(S) \cong H_\bullet(M(\xi), s_0(\mathbb{S}^m))$ (here we use that $s_0 : \mathbb{S}^m \longrightarrow M(\xi)$ is a cofibration). Therefore, S is a

1-connected homology $2m$-sphere, and thus, by the proof of the generalized Poincaré conjecture in higher dimensions, a sphere (in dimension 4, see Freedman [19], and in higher dimension Smale [54] and Newman [47]). In particular, $E_0 \cup \{o\} \cong \mathbb{R}^{2m}$ is an open cell. Thus, $M(\xi)$ is a compactification of an open $2m$-cell by an m-sphere.

Proposition 2.8. *Each model $M(\xi)$ can be decomposed as*

$$M(\xi) = X \mathbin{\dot\cup} U,$$

with $X = s_0(\mathbb{S}^m)$ homeomorphic to $\mathbb{S}^m$, and $U = E_0 \cup \{o\}$ open, dense, and homeomorphic to $\mathbb{R}^{2m}$.

3. Homeomorphisms between different models

In the last section, we constructed for every oriented $\mathbb{R}^m$-bundle ξ over $\mathbb{S}^m$ with absolute Euler number $|e| = |\langle e(\xi), [\mathbb{S}^m]\rangle| = 1$ a manifold $M(\xi)$. In this section, we determine under which conditions there are homeomorphisms $M(\xi) \cong M(\xi')$ between different models. Clearly, a weak bundle equivalence $\xi \cong_g \xi'$ induces a homeomorphism $M(\xi) \cong M(\xi')$ between the Thom spaces. We will see that this is in fact the only possibility. In the first part of this section, we assume only that m is even; in the second part, we return to the special case of our models where $m = 2, 4, 8$ and $|e| = 1$.

Let ξ be an $\mathbb{R}^m$-bundle over $\mathbb{S}^m$, with absolute Euler number $|e|$, for $m \geq 2$ even. Let $X = s_0(\mathbb{S}^m) \subseteq E$. By Proposition 1.12, the tangent bundle τE of the manifold E splits along X as a sum of a horizontal and a vertical bundle. Since $\tau\mathbb{S}^m \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{m+1}$, we have the following result.

Lemma 3.1. *Let τE denote the topological tangent bundle of E, and let $X = s_0(\mathbb{S}^m)$. Then $X \subseteq E$ is an embedded submanifold with normal bundle weakly equivalent to ξ and*

$$\xi \oplus \tau\mathbb{S}^m \cong s_0^*\tau E.$$

In particular, ξ and $\tau E|_X$ are weakly stably equivalent,

$$\xi \oplus \underline{\mathbb{R}}^{m+1} \cong s_0^*\tau E \oplus \underline{\mathbb{R}}.$$

Suppose now that there is a homeomorphism $E \xrightarrow[\cong]{f} E'$ of total spaces of $\mathbb{R}^m$ bundles ξ, ξ' over $\mathbb{S}^m$, for $m \geq 2$ even. Both s_0 and $f^{-1}s_0'$

represent generators of $\pi_m(E) \cong \mathbb{Z}$. Thus, there exists a homeomorphism $g : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ of degree ± 1 such that the diagram

$$\begin{tikzcd} E \arrow[r, "f", "\cong"'] & E' \\ \mathbb{S}^m \arrow[u, "s_0"] \arrow[r, "g", "\cong"'] & \mathbb{S}^m \arrow[u, "s_0'"'] \end{tikzcd}$$

is homotopy commutative. By Lemma 1.6, there is an equivalence $s_0^*\tau E \cong g^*(s_0')^*\tau E'$, whence $\xi \oplus \underline{\mathbb{R}}^{m+1} \cong g^*\xi' \oplus \underline{\mathbb{R}}^{m+1}$. In other words, the bundles ξ and ξ' are weakly stably equivalent.

Lemma 3.2. *Suppose that there is a homeomorphism of Thom spaces $M(\xi) \cong M(\xi')$. Then there is a homeomorphism between the total spaces $E \cong E'$, and the bundles ξ, ξ' have the same absolute Euler number.*

Proof. By Lemma 2.1, the absolute Euler number can be seen from the cohomology ring of $M(\xi)$, so $|e| = |e'|$. If $M(\xi)$ and $M(\xi')$ are manifolds, then they are homogeneous and the existence of a homeomorphism $M(\xi) \cong M(\xi')$ implies the existence of a homeomorphism $E \cong E'$. If $M(\xi)$ and $M(\xi')$ are not manifolds, then a homeomorphism maps the unique non-manifold point o of $M(\xi)$ onto the unique non-manifold point o' of $M(\xi')$, and so it maps E onto E'. q.e.d.

The proof of the next proposition involves classifying spaces, so we postpone it to 6.3.

Proposition 3.3. *Let ξ, ξ' be $\mathbb{R}^m$-bundles over $\mathbb{S}^m$, for $m \geq 2$ even. Suppose that there is a stable equivalence $\xi \oplus \underline{\mathbb{R}}^k \cong \xi' \oplus \underline{\mathbb{R}}^k$, and that the absolute Euler numbers of ξ and ξ' are equal, $|e| = |e'|$. Then there is an equivalence $\xi \cong \xi'$.*

Combining these results, we obtain a complete homeomorphism classification of the Thom spaces $M(\xi)$, for $m \geq 2$ even, in terms of bundles.

Proposition 3.4. *Let $m \geq 2$ be even, let ξ, ξ' be $\mathbb{R}^m$-bundles over $\mathbb{S}^m$. If there is a homeomorphism between the Thom spaces $M(\xi) \cong M(\xi')$, then ξ and ξ' are weakly equivalent.*

Proof. By Lemma 3.2 above, the total spaces E, E' are homeomorphic, and $|e| = |e'|$. The remarks at the begin of this section show that there is a weak equivalence between $\xi \oplus \underline{\mathbb{R}}^{m+1}$ and $\xi' \oplus \underline{\mathbb{R}}^{m+1}$, induced by a homeomorphism $g : \mathbb{S}^m \longrightarrow \mathbb{S}^m$. If $\deg(g) = 1$, then g is homotopic to the identity, whence $\xi \oplus \underline{\mathbb{R}}^{m+1} \cong \xi' \oplus \underline{\mathbb{R}}^{m+1}$. By Proposition 3.3, this implies that there is an equivalence $\xi \cong \xi'$. Otherwise, $\deg(g) = -1$ and we put $\xi'' = g^*\xi'$. Then we have an equivalence $\xi \oplus \underline{\mathbb{R}}^{m+1} \cong \xi'' \oplus \underline{\mathbb{R}}^{m+1}$, and, again by Proposition 3.3, an equivalence $\xi \cong \xi''$. But ξ' is weakly equivalent to ξ''. q.e.d.

Thus we have reduced the homeomorphism classification of our models to a classification of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$. For the specific values $m = 2, 4, 8$, this classification will be carried out in the next section. We end this section with some simple remarks about characteristic classes of our models. Each model $M(\xi)$ has a distinguished orientation: if y_m is any generator of $H^m(M(\xi))$, then y_m^2 is a generator of $H^{2m}(M(\xi))$ which does not depend on the choice of y_m. So we choose for our models the fundamental class $[M(\xi)] \in H^{2m}(M(\xi))$ in such a way that

$$\langle y_m^2, [M(\xi)]\rangle = 1.$$

Obviously, the Euler characteristic of any model $M(\xi)$ is

$$\chi_{M(\xi)} = 3.$$

Note also that any homeomorphism $c : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ of degree -1 induces a homeomorphism f_c of $M(\xi)$ with the property that $f_c^\bullet y_m = -y_m$. Thus every graded automorphism of the cohomology ring $H^\bullet(M(\xi)) \cong \mathbb{Z}[y_m]/(y_m^3)$ is induced by a homeomorphism.

Recall that the *Wu classes* $v_i \in H^i(M; \mathbb{Z}/2)$ of a closed manifold M are defined by $\langle v_i \smile x, [M]\rangle = \langle \mathrm{Sq}^i x, [M]\rangle$.

Lemma 3.5. *The total Stiefel-Whitney class of any model $M(\xi)$ is given by*

$$w(M(\xi)) = 1 + y_m + y_m^2,$$

where $y_m \in H^m(M(\xi); \mathbb{Z}/2)$ is a generator. Thus the minimal codimension for an embedding of $M(\xi)$ in $\mathbb{S}^{2m+k}$ or $\mathbb{R}^{2m+k}$ is $k = m + 1$.

Proof. We have $\mathrm{Sq}^m y_m = y_m^2$, so the total Wu class of $M(\xi)$ is $v = 1 + y_m + y_m^2$, and the total Stiefel-Whitney class is $w(M(\xi)) = \mathrm{Sq}\, v = 1 + y_m + y_m^2$ (see Spanier [55] Ch. 6.10 Theorem 7 and 6.10 8). The non-embedding result follows as in Spanier [55] Ch. 6.10 24. q.e.d.

Recall from 1.7 that the composite $\mathbb{S}^m \xrightarrow{s_0} X \subseteq E$ is a homotopy equivalence. Since E_0 is $2(m-1)$-connected, the exact homology sequence of the pair (E, E_0) shows that $H_m(E) \longrightarrow H_m(E, E_0)$ is an isomorphism.

Lemma 3.6. *The map $s_0 : \mathbb{S}^m \longrightarrow M(\xi)$ induces an isomorphism on the homotopy and* (*co*)*homology groups up to* (*and including*) *dimension m.*

Proof. Since $M(\xi)$ is 1-connected, the claim on the homotopy groups follows from the corresponding result for the homology groups by the Hurewicz isomorphism. q.e.d.

3.7. In particular, if $\gamma(M)$ is a stable m-dimensional characteristic class of the tangent bundle of $M(\xi)$ then $s_0^\bullet(\gamma(M)) = \gamma(\xi)$. For $m = 4, 8$, this applies in particular to the m-dimensional rational Pontrjagin class p_m of (the tangent bundle of) $M(\xi)$,

$$p_m(\xi) = s_0^\bullet p_m(M(\xi)),$$

so the m-dimensional Pontrjagin class of ξ determines the m-dimensional Pontrjagin class of $M(\xi)$. A similar result holds for the exotic classes, the Kirby-Siebenmann class ks and the class κ of M, which are constructed in 4.13. This will be used in the Section 4.

4. Characteristic classes

To get further, we need some results about classifying spaces. We refer to the books by Milnor-Stasheff [46], Kirby-Siebenmann [33], Madsen-Milgram [41] and to Ch. IV in Rudyak [50]. We denote the orthogonal group by $\mathrm{O}(n)$, and by $\mathrm{TOP}(n)$ the group of all base-point preserving homeomorphisms of $\mathbb{R}^n$. The corresponding classifying spaces are $\mathrm{BO}(n)$ and $\mathrm{BTOP}(n)$. If X is any space, then the set of free homotopy classes $[X; \mathrm{BO}(n)]$ is in one to one correspondence with the equivalence classes of numerable n-dimensional vector bundles over X. Similarly, $\mathrm{BTOP}(n)$ classifies numerable $\mathbb{R}^n$-bundles over X. Taking the limit $n \gg 1$ one obtains stable versions O, TOP of these groups; there are corresponding classifying spaces BO and BTOP which classify $\mathbb{R}^n$-bundles up to stable equivalence. We need two more classifying spaces. The space $\mathrm{BPL}(n)$ classifies $\mathbb{R}^n$-bundles (over simplicial complexes) which admit PL (piecewise linear) coordinate charts. To us, the main purpose of $\mathrm{BPL}(n)$ and its stable version BPL will be the fact that it lies somewhat in the middle between BTOP and BO. Finally, let $\mathrm{G}(n)$ denote the semigroup of all self-equivalences of the sphere $\mathbb{S}^{n-1}$. The corresponding classifying space $\mathrm{BG}(n)$ classifies $n-1$-spherical fibrations up to fiber homotopy equivalence. There are 1-connected coverings of these spaces which classify oriented bundles and fibrations: for example, the classifying space $\mathrm{BSO}(n)$ classifies n-dimensional oriented vector bundles. For $k \geq 0$ there are ladders of maps

$$\begin{tikzcd}
& \mathrm{BO}(n) \arrow[rr] \arrow[dd] & & \mathrm{BO}(n+k) \arrow[rr] \arrow[dd] & & \mathrm{BO} \arrow[dd] \\
\mathrm{BSO}(n) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{BSO}(n+k) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{BSO} \arrow[ur] \arrow[dd] & \\
& \mathrm{BPL}(n) \arrow[rr] \arrow[dd] & & \mathrm{BPL}(n+k) \arrow[rr] \arrow[dd] & & \mathrm{BPL} \arrow[dd] \\
\mathrm{BSPL}(n) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{BSPL}(n+k) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{BSPL} \arrow[ur] \arrow[dd] & \\
& \mathrm{BTOP}(n) \arrow[rr] \arrow[dd] & & \mathrm{BTOP}(n+k) \arrow[rr] \arrow[dd] & & \mathrm{BTOP} \arrow[dd] \\
\mathrm{BSTOP}(n) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{BSTOP}(n+k) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{BSTOP} \arrow[ur] \arrow[dd] & \\
& \mathrm{BG}(n) \arrow[rr] & & \mathrm{BG}(n+k) \arrow[rr] & & \mathrm{BG} \\
\mathrm{BSG}(n) \arrow[ur] \arrow[rr] & & \mathrm{BSG}(n+k) \arrow[ur] \arrow[rr] & & \mathrm{BSG} \arrow[ur] &
\end{tikzcd}$$

such that the diagram commutes (at least up to homotopy). The horizontal arrows correspond to the process of stabilization, i.e., if $f : X \longrightarrow \mathrm{BSTOP}(n)$ classifies ξ, then the composite

$$X \longrightarrow \mathrm{BSTOP}(n) \longrightarrow \mathrm{BSTOP}(n+k)$$

classifies $\xi \oplus \underline{\mathbb{R}}^k$. The vertical and the slanted arrows are 'forgetful': they forget the vector space structure, the PL structure, and the fiber bundle structure, respectively, and the slanted arrows forget the orientation. For these results, see e.g., Rudyak [50] Ch. IV.

By a well-known construction, every continuous map $f : X \longrightarrow Y$ between topological spaces can be converted into a fibration $f' : P_f \longrightarrow Y$, with $X \simeq P_f$, see Spanier [55] Ch. 2.8 Theorem 9. If Y is path-connected, then all fibers of f' have the same homotopy type, and it makes sense to speak about the homotopy fiber of the map f. Let $\mathrm{B}H \longrightarrow \mathrm{B}G$ be one of the maps in the diagram above. The homotopy fiber of this map is denoted by G/H. One can show that the homotopy fiber of the map $\mathrm{BO}(n) \longrightarrow \mathrm{BO}(n+k)$ is homotopy equivalent to the Stiefel manifold $\mathrm{O}(n+k)/\mathrm{O}(n)$, so this terminology fits together with the standard Lie group terminology.

From the homotopy viewpoint, the stable classifying spaces are much easier to understand. This is partly due to the fact that they are H-spaces (the Whitney sum of bundles is the multiplication). The homotopy groups of BO are known by Bott periodicity, $\mathrm{BO} \times \mathbb{Z} \simeq \Omega^8 \mathrm{BO}$, see Bott [4] [5], i.e.,

$$\pi_k(\mathrm{BO} \times \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k \equiv 0 \pmod 4 \\ \mathbb{Z}/2 & \text{for } k \equiv 1, 2 \pmod 8 \\ 0 & \text{else.} \end{cases}$$

The homotopy groups of BG correspond to the stable homotopy groups of spheres, see, e.g., Milnor [44] §2 and Madsen-Milgram [41] Ch. 3,

$$\pi_{k+1}(\mathrm{BG}) \cong \lim_{n \to \infty} \pi_{k+n}(\mathbb{S}^n) = \pi_k^s(\mathbb{S}^0)$$

which are known in low dimensions, see, e.g., Toda [60] Ch. XIV, Hu [27] pp. 328–332, or Fomenko-Fuchs-Gutenmacher [18] pp. 300–301. For the other homotopy fibers, we use the following results which are obtained from surgery theory.

Theorem 4.1. *The homotopy groups of* $\mathrm{G/TOP}$ *are given by the periodicity* $\mathrm{G/TOP} \times \mathbb{Z} \simeq \Omega^4(\mathrm{G/TOP})$, *see Kirby-Siebenmann* [33] *p.* 327 (*the* $\mathbb{Z}$*-factor is forgotten there*), *and*

$$\pi_k(\mathrm{G/TOP} \times \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k \equiv 0 \pmod 4 \\ \mathbb{Z}/2 & \text{for } k \equiv 2 \pmod 4 \\ 0 & \text{else,} \end{cases}$$

see also Madsen-Milgram [41] Ch. 2.

Finally, we use the following result (see Kirby-Siebenmann [33, p. 200]).

Theorem 4.2. *The homotopy groups of* $\mathrm{TOP/O}$ *and* $\mathrm{TOP/PL}$ *are finite in all dimensions. If* $i \geq 5$, *then* $\pi_i(\mathrm{TOP/O})$ *is isomorphic to the Kervaire-Milnor group* Θ_i *of* DIFF *structures on* $\mathbb{S}^i$ (*see Kirby-Siebenmann* [33] p. 200, 251), *and* $\mathrm{TOP/PL}$ *is an Eilenberg-MacLane space of type* $K(\mathbb{Z}/2, 3)$.

It follows from Serre's $\mathcal{C}$-theory that

$$\widetilde{H}^\bullet(\mathrm{TOP}/\mathrm{O};\mathbb{Q}) \cong \widetilde{H}^\bullet(\mathrm{TOP}/\mathrm{PL};\mathbb{Q}) \cong \widetilde{H}^\bullet(\mathrm{PL}/\mathrm{O};\mathbb{Q}) \cong 0$$

(see Spanier [55] Ch. 9.6). The Serre spectral sequence yields thus natural isomorphisms

$$H^\bullet(\mathrm{BO};\mathbb{Q}) \xleftarrow{\cong} H^\bullet(\mathrm{BPL};\mathbb{Q}) \xleftarrow{\cong} H^\bullet(\mathrm{BTOP};\mathbb{Q}),$$

see also Kahn [31]. The rational cohomology ring of BO is known to be a polynomial algebra generated by the *universal Pontrjagin classes* $p_4, p_8, \dots,$

$$H^\bullet(\mathrm{BO};\mathbb{Q}) \cong \mathbb{Q}[p_4, p_8, p_{12}, \dots],$$

see e.g., Madsen-Milgram [41] p. 13. In view of the isomorphisms above, there are natural rational Pontrjagin classes defined for $\mathbb{R}^n$-bundles and for PL-bundles: if $X \longrightarrow \mathrm{BTOP}(n)$ is a classifying map for an $\mathbb{R}^n$-bundle ξ, then $p_{4k}(\xi) \in H^{4k}(X;\mathbb{Q})$ is by definition the pull-back of the universal Pontrjagin class $p_{4k} \in H^{4k}(\mathrm{BTOP};\mathbb{Q})$ via the composite $X \longrightarrow \mathrm{BTOP}(n) \longrightarrow \mathrm{BTOP}$, see also Milnor-Stasheff [46] pp. 250–251.

4.3. Recall that the *signature* of an oriented $4k$-manifold M is by definition the signature of the quadratic form $H^{2k}(M;\mathbb{Q}) \longrightarrow \mathbb{Q}$, $v \longmapsto \langle v^2, [M]\rangle$. (The signature of a rational quadratic form represented by a symmetric matrix is the number of strictly positive eigenvalues minus the number of strictly negative real eigenvalues of the matrix.) The signature is invariant under oriented bordism and induces a group homomorphism from the oriented DIFF bordism ring into the integers,

$$\mathrm{Sig} : \Omega_\bullet^{\mathrm{SO}} \longrightarrow \mathbb{Z}.$$

It follows that the signature of a smooth closed $4k$-manifold can be expressed as a certain linear combination of the rational Pontrjagin numbers of M which is given by Hirzebruch's $\mathcal{L}$-genus,

$$\mathrm{Sig}(M) = \langle \mathcal{L}_{4k}(M), [M]\rangle,$$

see Milnor-Stasheff [46] p. 224 and Madsen-Milgram [41] Theorem 1.38. The *Hirzebruch classes* $\mathcal{L}_{4k} \in H^{4k}(\mathrm{BO};\mathbb{Q})$ are certain rational homogeneous polynomials in the rational Pontrjagin classes, see Hirzebruch [24] 1.5, Milnor-Stasheff [46] §19. These polynomials can be obtained by a formal process from the power series expansion of the function $f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$. These results were proved by Hirzebruch for smooth closed oriented manifolds, see Hirzebruch [24] Hauptsatz 8.2.2. However, there is an isomorphism

$$\Omega_\bullet^{\mathrm{SO}} \otimes \mathbb{Q} \cong \Omega_\bullet^{\mathrm{STOP}} \otimes \mathbb{Q},$$

so the Signature Theorem carries over to the bordism ring of closed oriented manifolds, see Kahn [31], Kirby-Siebenmann [33] p. 322.

Theorem 4.4. *Let M be a closed oriented $4k$-manifold with fundamental class $[M]$. Then*

$$\mathrm{Sig}(M) = \langle \mathcal{L}_{4k}(M), [M] \rangle.$$

The signature of our models is clearly $\mathrm{Sig}(M(\xi)) = 1$.

Lemma 4.5. *For our models, $M(\xi)$, we have the following relation for the Hirzebruch classes:*

$$\mathcal{L}_{2m}(M(\xi)) = y_m^2.$$

To compute the Hirzebruch classes in terms of the Pontrjagin classes, we use the following result which is a direct consequence of Hirzebruch [24] 1.4. Let $f(t) = 1 + \sum_{k\geq 1} f_k t^k$ be a formal power series, and let $\{K_k(\sigma_1, \ldots, \sigma_k)\}_{k=1}^\infty$ denote the corresponding multiplicative sequence, see Hirzebruch [23] §1. The associated genus $\mathcal{F}$ of $f(t)$ is defined by

$$\mathcal{F}_{4k} = K_k(p_4, \ldots, p_{4k}) \in H^{4k}(\mathrm{BO}; \mathbb{Q}).$$

Hirzebruch's $\mathcal{L}$-genus comes from the formal power series

$$\ell(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \frac{2}{945}t^3 - \frac{1}{4725}t^4 + \cdots,$$

and the $\widehat{\mathcal{A}}$-genus (which will be needed later in Section 7) from

$$\begin{aligned}\hat{a}(t) &= \frac{\sqrt{t}/2}{\sinh\sqrt{t}/2} \\ &= 1 - \frac{1}{24}t + \frac{7}{5760}t^2 - \frac{31}{967680}t^3 + \frac{127}{154828800}t^4 + \cdots.\end{aligned}$$

Suppose now that ξ is an $\mathbb{R}^n$ bundle over a space X, and that $p_{4k}(\xi)$ and $p_{8k}(\xi)$ are the only nonzero Pontrjagin classes of ξ. This holds for example if X is a space with $H^j(X;\mathbb{Q}) = 0$ for all $j \neq 0, 4k, 8k$ (such as our models $M(\xi)$). In the cohomology ring of such a space, there is the following general formula for $\mathcal{F}_{8k} = K_{2k}(0, \cdots, 0, p_{4k}, 0, \ldots, 0, p_{8k})$. Put

$$f^\vee(t) = f(t)\frac{d}{dt}\left(\frac{t}{f(t)}\right) = \sum(-1)^{\mathrm{ks}}_k t^k.$$

By *loc.cit.* 1.4, one has the general relation

$$K_{2k}(0, \ldots, 0, p_{4k}, 0, \ldots, p_{8k}) = s_{2k}p_{8k} + \frac{1}{2}(s_k^2 - s_{2k})p_{4k}^2$$

For $\ell^\vee$ and $\hat{a}^\vee$ one obtains

$$\begin{aligned}
\ell^\vee(t) &= 1 - \frac{1}{3}t + \frac{7}{45}t^2 - \frac{62}{945}t^3 + \frac{127}{4725}t^4 + \cdots \\
\hat{a}^\vee(t) &= 1 + \frac{1}{24}t - \frac{1}{1440}t^2 + \frac{1}{60480}t^3 - \frac{1}{2419200}t^4 + \cdots .
\end{aligned}$$

4.6. For specific values, calculations can readily be done with the formal power series package of Maple. In low dimensions, one obtains the following.

$$\begin{aligned}
\mathcal{L}_4(p_4) &= \frac{1}{3}p_4 \\
\widehat{\mathcal{A}}_4(p_4) &= \frac{-1}{2^3 \cdot 3}p_4 \\
\mathcal{L}_8(p_4, p_8) &= \frac{1}{3^2 \cdot 5}(7p_8 - p_4^2) \\
\widehat{\mathcal{A}}_8(p_4, p_8) &= \frac{-1}{2^7 \cdot 3^2 \cdot 5}(4p_8 - 7p_4^2) \\
\mathcal{L}_{16}(0, p_8, 0, p_{16}) &= \frac{1}{3^4 \cdot 5^2 \cdot 7}(381p_{16} - 19p_8^2) \\
\widehat{\mathcal{A}}_{16}(0, p_8, 0, p_{16}) &= \frac{-1}{2^{11} \cdot 3^4 \cdot 5^2 \cdot 7}(12p_{16} - 13p_8^2).
\end{aligned}$$

In the last two equations, we assume thus that $p_4 = 0 = p_{12}$. See also Hirzebruch [24] 1.5 and 1.6, Milnor-Stasheff [46] p. 225, Lawson-Michelsohn [39] pp. 231–232, Eells-Kuiper [15] p. 105 for explicit formulas for these genera in low dimensions.

From the signature theorem, we obtain thus for our models strong relations between p_m and p_{2m}.

Lemma 4.7. *For our models, we have*

$$
\begin{aligned}
p_4(M) &= \frac{1}{3}y_2^2 && (m=2)\\
p_8(M) &= \frac{1}{7}(45y_4^2 + p_4(M)^2) && (m=4)\\
p_{16}(M) &= \frac{1}{381}(14175y_8^2 + 19p_8(M)^2) && (m=8).
\end{aligned}
$$

4.8. We need also the first exotic characteristic classes for TOP- and PL-bundles. By a standard mapping cylinder construction, we may convert the maps $\mathrm{BO} \longrightarrow \mathrm{BPL} \longrightarrow \mathrm{BTOP}$ into cofibrations, see Spanier [55] Ch. 3.2 Theorem 12. Thus, it makes sense to speak of the homology and homotopy groups of the pairs $(\mathrm{BTOP}, \mathrm{BO})$ etc. Note also that these three spaces are H-spaces with isomorphic fundamental groups. Thus, $\pi_1(\mathrm{BO}) \cong \pi_1(\mathrm{BPL}) \cong \pi_1(\mathrm{BTOP}) \cong \mathbb{Z}/2$ acts trivially on the homotopy groups of each of these pairs, whence $\pi_k = \pi'_k$ for these pairs (recall that π'_k is the kth homotopy group, factored by the action of π_1, see Spanier [55] p. 390). Consequently, we have Hurewicz isomorphisms $H_k(\mathrm{BPL}, \mathrm{BO}) \cong \pi_k(\mathrm{BPL}, \mathrm{BO})$ up to and including the lowest dimensions where the right-hand side is nontrivial, see Spanier [55] Ch. 7.5 Theorem 4. Now there is an isomorphism $\pi_{k-1}(\mathrm{PL/O}) \cong \pi_k(\mathrm{BPL}, \mathrm{BO})$, see Whitehead [66] Ch. IV 8.20. In fact, there is a commutative diagram

$$
\begin{tikzcd}
\pi_{k-1}(\mathrm{PL/O}) \arrow[d] & \pi_k(C_{\mathrm{PL/O}}, \mathrm{PL/O}) \arrow[l, "\partial"', "\cong"] \arrow[r, "\cong"'] \arrow[d] & \pi_k(\mathrm{BPL}, \mathrm{BO}) \arrow[d] \\
H_{k-1}(\mathrm{PL/O}) & H_k(C_{\mathrm{PL/O}}, \mathrm{PL/O}) \arrow[l, "\partial"', "\cong"] \arrow[r] & H_k(\mathrm{BPL}, \mathrm{BO})
\end{tikzcd}
$$

where $C_{\mathrm{PL/O}} \simeq *$ is the unreduced cone. Since PL/O is 6-connected, see Madsen-Milgram p. 33, this diagram consists of isomorphisms for $k \leq 8$. We obtain a similar diagram for TOP/O and TOP/PL, with isomorphisms for $k \leq 4$, using the fact that TOP/PL and TOP/O are 2-connected, see Kirby-Siebenmann p. 246.

We combine this with the following result due to Hirsch [23] p. 356.

Proposition 4.9. *There are short exact sequences*

$$0 \longrightarrow \pi_k(\mathrm{BO}) \longrightarrow \pi_k(\mathrm{BPL}) \longrightarrow \pi_k(\mathrm{BPL}, \mathrm{BO}) \longrightarrow 0$$

for all $k \geq 0$.

4.10. Since TOP/PL is an Eilenberg-MacLane space of type $K(\mathbb{Z}/2,3)$, and since PL/O is 6-connected, this implies readily that each arrow

$$\pi_k(\mathrm{BO}) \longrightarrow \pi_k(\mathrm{BPL}) \longrightarrow \pi_k(\mathrm{BTOP})$$

is an injection, for all $k \geq 0$.

We combine this with the commutative diagram in 4.8 and obtain thus the following result.

Lemma 4.11. *There are natural isomorphisms*

$$H_k(\mathrm{BO}) \xrightarrow{\cong} H_k(\mathrm{BPL}) \xrightarrow{\cong} H_k(\mathrm{BTOP})$$

for $k \leq 3$, *and* $H_k(\mathrm{BO}) \xrightarrow{\cong} H_k(\mathrm{BPL})$ *for* $k \leq 7$, *and exact sequences*

$$H_4(\mathrm{BO}) \longrightarrow H_4(\mathrm{BTOP}) \longrightarrow H_4(\mathrm{BTOP},\mathrm{BO}) \longrightarrow 0$$

$$H_4(\mathrm{BPL}) \longrightarrow H_4(\mathrm{BTOP}) \longrightarrow H_4(\mathrm{BTOP},\mathrm{BPL}) \longrightarrow 0$$

$$H_8(\mathrm{BO}) \longrightarrow H_8(\mathrm{BPL}) \longrightarrow H_8(\mathrm{BPL},\mathrm{BO}) \longrightarrow 0.$$

Furthermore, $H_4(\mathrm{BTOP},\mathrm{BO}) \cong H_4(\mathrm{BTOP},\mathrm{BPL}) \cong \mathbb{Z}/2$ *and* $H_8(\mathrm{BPL},\mathrm{BO}) \cong \mathbb{Z}/28$.

Proof. The corresponding homotopy groups are given in Kirby-Siebenmann [33] p. 246. q.e.d.

From the universal coefficient theorem, see Spanier [55] Ch. 5.5 Theorem 3, we have the following result.

Proposition 4.12. *For any coefficient domain* R, *there are exact sequences*

$$H^4(\mathrm{BO};R) \longleftarrow H^4(\mathrm{BTOP};R) \xleftarrow{\tau_{\mathrm{TOP/O}}} H^3(\mathrm{TOP/O};R) \longleftarrow 0$$

$$H^4(\mathrm{BPL};R) \longleftarrow H^4(\mathrm{BTOP};R) \xleftarrow{\tau_{\mathrm{TOP/PL}}} H^3(\mathrm{TOP/PL};R) \longleftarrow 0$$

$$H^8(\mathrm{BO};R) \longleftarrow H^8(\mathrm{BPL};R) \xleftarrow{\tau_{\mathrm{PL/O}}} H^7(\mathrm{PL/O};R) \longleftarrow 0.$$

4.13. The $\tau_{\mathrm{TOP/PL}}$-image of the generator of $H^3(\mathrm{TOP/PL};\mathbb{Z}/2) \cong \mathbb{Z}/2$ is the universal *Kirby-Siebenmann class* $\mathrm{ks} \in H^4(\mathrm{BTOP};\mathbb{Z}/2)$. For

$R = \mathbb{Z}/4$, we pick a generator κ of the $\tau_{\mathrm{PL/O}}$-image of $H^7(\mathrm{PL/O};\mathbb{Z}/4) \cong \mathbb{Z}/4$. Thus we define the first exotic characteristic classes

$$\mathrm{ks} \in H^4(\mathrm{BTOP};\mathbb{Z}/2) \quad \text{and} \quad \kappa \in H^8(\mathrm{BPL};\mathbb{Z}/4).$$

The map τ is the transgression (see McCleary [42] 6.2). This is maybe not obvious from our construction. To see this, consider the following diagram of cochain complexes.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & S^\bullet(\mathrm{PL/O}) \arrow[l] \arrow[u] & S^\bullet(\mathrm{BO}) \arrow[l] \arrow[u] & S^\bullet(\mathrm{BO},\mathrm{PL/O}) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & S^\bullet(C_{\mathrm{PL/O}}) \arrow[l] \arrow[u] & S^\bullet(\mathrm{BPL}) \arrow[l] \arrow[u] & S^\bullet(\mathrm{BPL},C_{\mathrm{PL/O}}) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & S^\bullet(C_{\mathrm{PL/O}},\mathrm{PL/O}) \arrow[l] \arrow[u] & S^\bullet(\mathrm{BPL},\mathrm{BO}) \arrow[l] \arrow[u] & S^\bullet(\mathrm{BPL},\mathrm{BO},C_{\mathrm{PL/O}}) \arrow[l] \arrow[u] & 0 \arrow[l] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here, $S^\bullet(\mathrm{BPL},\mathrm{BO},C_{\mathrm{PL/O}})$ denotes the singular cochain complex of the triad $(\mathrm{BPL},\mathrm{BO},C_{\mathrm{PL/O}})$, see Eilenberg-Steenrod [16] VII.11. Using the isomorphisms derived above, patient diagram chasing in the corresponding big diagram for cohomology (the infinite cohomology jail window, see, e.g., Cartan-Eilenberg [10] IV Proposition 2.1) shows that τ is indeed the transgression.

5. Stable $\mathbb{R}^n$-bundles over $\mathbb{S}^m$

Our aim is the classification of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$ in terms of characteristic classes. We begin with the stable classification, which is easier. Recall from 4.10 that there is an exact sequence

$$0 \longrightarrow \pi_k(\mathrm{BO}) \longrightarrow \pi_k(\mathrm{BTOP}) \longrightarrow \pi_{k-1}(\mathrm{TOP/O}) \longrightarrow 0$$

for all $k \geq 0$.

Lemma 5.1. *In dimensions $k = 2, 4, 8$ these exact sequences read* *as follows.*

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0 & (k=2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 & (k=4)
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}/4 \arrow[r] & \mathbb{Z}/28 \arrow[r] & 0 & (k=8).
\end{tikzcd}
$$

Proof. We consider the maps

$$
\begin{tikzcd}
 & & \mathrm{G/TOP} \arrow[d] \\
\mathrm{TOP/O} \arrow[r] & \mathrm{BO} \arrow[r] & \mathrm{BTOP} \arrow[d] \\
 & & \mathrm{BG}.
\end{tikzcd}
$$

The space TOP/O is 2-connected, see Kirby-Siebenmann [33] p. 246, and this establishes the result for $m = 2$. Furthermore $\pi_3(\mathrm{TOP/O}) \cong \mathbb{Z}/2$ and $\pi_7(\mathrm{TOP/O}) \cong \mathbb{Z}/28$, see Kirby-Siebenmann [33] p. 246 and 200, and Kervaire-Milnor [32]. From the isomorphisms $\pi_k(\mathrm{BG}) \cong \pi^s_{k-1}(\mathbb{S}^0)$ we have $\pi_4(\mathrm{BG}) \cong \mathbb{Z}/24$ and $\pi_8(\mathrm{BG}) \cong \mathbb{Z}/240$, see Toda [60] Ch. XIV, Hu [27] Ch. XI. Theorem 16.4 and p. 332, or Fomenko-Fuchs-Gutenmacher [18] p. 300. Finally, $\pi_{4k}(\mathrm{BO}) \cong \mathbb{Z}$ for $k \geq 1$ by Bott periodicity. Thus we obtain diagrams

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[d, "\text{mono}"] & \\
\mathbb{Z} \arrow[r, "\text{mono}"] \arrow[dr, "\text{epi}", "J_{\mathrm{O}}"'] & \pi_4(\mathrm{BTOP}) \arrow[r, "\text{epi}"] \arrow[d, "\text{epi}"] & \mathbb{Z}/2 \\
 & \mathbb{Z}/24 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[d, "\text{mono}"] & \\
\mathbb{Z} \arrow[r, "\text{mono}"] \arrow[dr, "\text{epi}", "J_{\mathrm{O}}"'] & \pi_8(\mathrm{BTOP}) \arrow[r, "\text{epi}"] \arrow[d, "\text{epi}"] & \mathbb{Z}/28 \\
 & \mathbb{Z}/240. &
\end{tikzcd}
$$

In these diagrams, the rows are short exact sequences by the remarks at the beginning of this section. The columns are also short exact, since $\pi_{4k-1}(\mathrm{G/TOP}) = 0$, while $\pi_{4k+1}(\mathrm{BG})$ is finite. The slanted arrows J_{O} are known to be epimorphisms in these dimensions, see e.g., Adams [1] p. 22 and p. 46.

Let T_{4k} denote the torsion group of $\pi_{4k}(\mathrm{BTOP})$, i.e., $\pi_{4k}(\mathrm{BTOP}) \cong \mathbb{Z} \oplus T_{4k}$. Suppose that $m = 4$. The diagram shows that T_4 injects into $\mathbb{Z}/2$. If we tensor the diagram with $\mathbb{Z}/2$, then $J_{\mathrm{O}} \otimes \mathbb{Z}/2$ is a bijection. Therefore, the horizontal sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \oplus (T_4 \otimes \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

is still exact after tensoring (the only point to check is injectivity of the second arrow). It follows that $T_4 \cong \mathbb{Z}/2$.

The case $m = 8$ is similar. The group T_8 injects into $\mathbb{Z}/28$ and into $\mathbb{Z}/240$. Since $\gcd(28, 240) = 4$, the group injects into $\mathbb{Z}/4$. Similarly as in the case $m = 4$, tensoring the diagram with $\mathbb{Z}/4$ we see that $J_{\mathrm{O}} \otimes \mathbb{Z}/4$ is an isomorphism. Thus the sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4 \oplus (T_8 \otimes \mathbb{Z}/4) \longrightarrow \mathbb{Z}/4 \longrightarrow 0$$

is exact, and $T_8 \cong \mathbb{Z}/4$. q.e.d.

For $k \geq 3$, the structure of the torsion groups T_{4k} was determined by Brumfiel [8], see Madsen-Milgram [41] p. 117. The cases $k = 1, 2$ are special; they are considered in Kirby-Siebenmann [33] p. 318 and Williamson [67] p. 29.

5.2. The result above yields thus exact sequences

$$0 \longrightarrow \pi_4(\mathrm{BO}) \xrightarrow{\cong} \pi_4(\mathrm{BTOP})/T_4 \longrightarrow 0 \longrightarrow 0 \quad (m = 4)$$

$$0 \longrightarrow \pi_8(\mathrm{BO}) \longrightarrow \pi_8(\mathrm{BTOP})/T_8 \longrightarrow \mathbb{Z}/7 \longrightarrow 0 \quad (m = 8).$$

The cokernels of the corresponding maps $\pi_{4k}(\mathrm{BO}) \longrightarrow \pi_{4k}(\mathrm{BTOP})/T_{4k}$ for $k \geq 3$ are determined in Brumfiel [8] p. 304 in number theoretic terms.

Consider now the map which assigns to a stable $\mathbb{R}^n$-bundle ξ over $\mathbb{S}^{4k}$ the rational Pontrjagin number $\langle p_{4k}(\xi), [\mathbb{S}^{4k}]\rangle$. We want to determine the possible values of this map, and we do this first for vector bundles.

5.3. For a finite connected CW-complex X and an n-dimensional vector bundle ξ over X, there is a classifying map $X \longrightarrow \mathrm{BO}(n)$. The Pontrjagin classes of X are obtained by pulling back the universal Pontrjagin classes in $H^\bullet(\mathrm{BO}; \mathbb{Q})$ via the composite $X \longrightarrow \mathrm{BO}(n) \longrightarrow \mathrm{BO}$. Only the homotopy type of this map is important, so we view it as an element of the set $[X; \mathrm{BO}]$ of free homotopy classes of maps from X to

BO (since BO is an H-space and X is connected, we have $[X;\mathrm{BO}]_0 = [X,\mathrm{BO}]$, as discussed in the remarks at the end of the introduction of this paper). Now this set $[X;\mathrm{BO}]$ can be identified with the real reduced $\widetilde{\mathrm{KO}}$-theory of X,

$$[X;\mathrm{BO}] = \widetilde{\mathrm{KO}}(X),$$

see Atiyah-Hirzebruch [3], Hirzebruch [25] or Husemoller [30]. The *Pontrjagin character* is the ring homomorphism $\mathrm{ph} = \mathrm{ch}\circ\mathrm{cplx}$,

$$\mathrm{KO}(X) \xrightarrow{\mathrm{cplx}} \mathrm{KU}(X) \xrightarrow{\mathrm{ch}} H^\bullet(X;\mathbb{Q})$$

where cplx denotes complexification of real vector bundles, and ch denotes the *Chern character* of complex KU-theory, see Atiyah-Hirzebruch [3] or Hirzebruch [25] 1.4. For the following facts see Hirzebruch [25] 1.4–1.6. Hirzebruch's integrality theorem says that

$$\mathrm{ch}(\mathrm{KU}(\mathbb{S}^{2k})) = H^\bullet(\mathbb{S}^{2k}) \subseteq H^\bullet(\mathbb{S}^{2k};\mathbb{Q}),$$

with $\mathrm{ch}(\eta) = \mathrm{rk}_{\mathbb{C}}(\eta) + (-1)^{k-1}\frac{1}{(k-1)!}c_k(\eta)$. Recall also that $p_{4k}(\xi) = (-1)^k c_{2k}(\xi\otimes\mathbb{C})$. Thus we have the formula

$$\mathrm{ph}(\xi) = \mathrm{rk}_{\mathbb{R}}(\xi) + (-1)^{k-1}\frac{1}{(2k-1)!}p_{4k}(\xi)$$

on $\mathbb{S}^{4k}$. The map $\widetilde{\mathrm{KO}}(\mathbb{S}^{4k}) \xrightarrow{\mathrm{cplx}} \widetilde{\mathrm{KU}}(\mathbb{S}^{4k})$ is injective, with cokernel 0 for k even, and cokernel $\mathbb{Z}/2$ for k odd.

Combining these facts, we have the following result.

Lemma 5.4. *Let ξ be a vector bundle over $\mathbb{S}^{4k}$, and let $x \in H^{4k}(\mathbb{S}^{4k})$ be a generator. Then*

$$p_{4k}(\xi) = a_\xi \cdot d_k \cdot (2k-1)! \cdot x$$

where a_ξ is an integer depending on ξ, and $d_k = 1$ for k even, $d_k = 2$ for k odd. Conversely, given any integer a_ξ, there exists a vector bundle ξ with such a Pontrjagin class, and ξ is unique up to stable equivalence.

Proof. Since BO is an H-space, the set $[\mathbb{S}^{4k};\mathrm{BO}]$ of free homotopy classes coincides with the homotopy group $\pi_{4k}(\mathrm{BO}) = [\mathbb{S}^{4k};\mathrm{BO}]_0$. Viewing the universal Pontrjagin class p_{4k} as a map $\mathrm{BTOP} \xrightarrow{p_{4k}} K(\mathbb{Q},4k)$

into an Eilenberg-MacLane space, we obtain a nontrivial group homomorphism

$$\mathbb{Z} \cong \pi_{4k}(\mathrm{BO}) \xrightarrow{(p_{4k})_\#} \pi_{4k}(K(\mathbb{Q}, 4k)) \cong \mathbb{Q}.$$

q.e.d.

If ξ is a vector bundle, the possible values of the rational numbers $\langle p_{4k}(\xi), [\mathbb{S}^m]\rangle$ are thus the integral multiples of $d_k \cdot (2k-1)!$.

Lemma 5.5. *Let ξ be an $\mathbb{R}^n$-bundle over $\mathbb{S}^m$, for $m = 4, 8$. Then*

$$\langle p_4(\xi), [\mathbb{S}^4]\rangle \in 2\mathbb{Z} \subseteq \mathbb{Q}$$
$$\langle p_8(\xi), [\mathbb{S}^8]\rangle \in \tfrac{6}{7}\mathbb{Z} \subseteq \mathbb{Q}.$$

Conversely, for each of these values, there exists an $\mathbb{R}^n$-bundle (for some sufficiently large n) whose Pontrjagin number assumes this value.

Proof. This is clear from Lemma 5.4, applied to the special cases $k = 1, 2$, and the formula for the cokernel of the map

$$\pi_{4k}(\mathrm{BO}) \longrightarrow \pi_{4k}(\mathrm{BTOP})/T_{4k}$$

which was derived in 5.2. q.e.d.

We have proved the following result.

Proposition 5.6. *Let ξ be an $\mathbb{R}^n$-bundle over $\mathbb{S}^m$, for $m = 2, 4, 8$. Up to stable equivalence, the bundle ξ is completely determined by the following characteristic classes:*

($m = 2$) *its 2nd Stiefel-Whitney class $w_2(\xi)$.*

($m = 4$) *its 4-dimensional Pontrjagin class $p_4(\xi)$ and its Kirby-Siebenmann class $\mathrm{ks}(\xi)$.*

($m = 8$) *its 8-dimensional Pontrjagin class $p_8(\xi)$ and the characteristic class $\kappa(\xi)$.*

The possible ranges for the values of these characteristic classes, evaluated on the fundamental class $[\mathbb{S}^m]$, are $\mathbb{Z}/2$ (for $m = 2$), $2\mathbb{Z}$ and $\mathbb{Z}/2$ (for $m = 4$), and $\frac{6}{7}\mathbb{Z}$ and $\mathbb{Z}/4$ (for $m = 4$), respectively.

Proof. We prove the 8-dimensional case; the others are similar. Let $(\mathrm{BPL}, \mathrm{BO}) \longrightarrow (K(\mathbb{Z}/4, 8), *)$ represent the generator of $H^8(\mathrm{BPL}, \mathrm{BO};$

$\mathbb{Z}/4)$ which maps to κ. Then the composite $\mathrm{BPL} \longrightarrow (\mathrm{BPL}, \mathrm{BO}) \longrightarrow (K(\mathbb{Z}/4, 8), *)$ induces an isomorphisms on the $\mathbb{Z}/4$-factors in

$$\pi_8(\mathrm{BPL}) \longrightarrow \pi_8(\mathrm{BPL}, \mathrm{BO}) \longrightarrow \pi_8(K(\mathbb{Z}/4, 8)),$$

which can be identified with the map $\xi \longmapsto \kappa(\xi) \in H^8(\mathrm{BPL}; \mathbb{Z}/4)$. The octonionic Hopf bundle $\eta_{\mathbb{O}}$ over $\mathbb{S}^8$ represents an element of $\pi_8(\mathrm{BPL})$ with $p_8(\eta_{\mathbb{O}}) \neq 0$. Thus, the map $\xi \mapsto (p_8(\xi), \kappa(\xi)) \in H^8(\mathbb{S}^8; \mathbb{Q}) \oplus H^8(\mathbb{S}^8; \mathbb{Z}/4) \cong \mathbb{Q} \oplus \mathbb{Z}/4$ is an injection (with image $\frac{6}{7}\mathbb{Z} \oplus \mathbb{Z}/4$). q.e.d.

6. $\mathbb{R}^m$-bundles over $\mathbb{S}^m$

In the previous section, we classified bundles over $\mathbb{S}^m$ up to stable equivalence in terms of characteristic classes. To obtain an unstable classification, i.e., a classification of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$, we use the following result.

Proposition 6.1. *Let $m \geq 2$ be even. Then there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & \pi_m(\mathbb{S}^m) \arrow[r] \arrow[d, equal] & \pi_m(\mathrm{BO}(m)) \arrow[r] \arrow[d] & \pi_m(\mathrm{BO}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_m(\mathbb{S}^m) \arrow[r] \arrow[d, equal] & \pi_m(\mathrm{BPL}(m)) \arrow[r] \arrow[d] & \pi_m(\mathrm{BPL}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_m(\mathbb{S}^m) \arrow[r] \arrow[d, equal] & \pi_m(\mathrm{BTOP}(m)) \arrow[r] \arrow[d] & \pi_m(\mathrm{BTOP}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_m(\mathbb{S}^m) \arrow[r] & \pi_m(\mathrm{BG}(m)) \arrow[r] & \pi_m(\mathrm{BG}) \arrow[r] & 0.
\end{tikzcd}$$

In this diagram, the second column of vertical arrows is induced by the respective classifying map of the tangent bundle of $\mathbb{S}^m$ (resp. its underlying spherical fibration) and the third column of vertical arrows corresponds to stabilization.

Proof. This is proved in [37] pp. 93–95; there, the result is stated for the oriented case, but the homotopy groups are the same. The PL result is not stated in [37], but in low dimensions, $\pi_k(\mathrm{BO}) \cong \pi_k(\mathrm{BPL})$ and in higher dimensions $\pi_k(\mathrm{BPL}) \cong \pi_k(\mathrm{BTOP})$, see Kirby-Siebenmann [33] V §5. For a related result (but excluding dimension 4) see Varadarajan [61]. q.e.d.

Our aim is to prove that the first three rows in this diagram split, provided that $m \geq 4$. It suffices to prove this for the third row; the diagram then implies the splitting of the first two rows.

Lemma 6.2. *Let $m \geq 4$ be even. Then the exact sequence*

$$0 \longrightarrow \pi_m(\mathbb{S}^m) \longrightarrow \pi_m(\mathrm{BTOP}(m)) \longrightarrow \pi_m(\mathrm{BTOP}) \longrightarrow 0$$

splits.

Proof. Since we consider only higher dimensional homotopy groups, we may as well consider the universal coverings $\mathrm{BSTOP}(m)$ and BSTOP which classify oriented bundles. So we have an exact sequence

$$0 \longrightarrow \pi_m(\mathbb{S}^m) \longrightarrow \pi_m(\mathrm{BSTOP}(m)) \longrightarrow \pi_m(\mathrm{BSTOP}) \longrightarrow 0.$$

Let e denote the universal Euler class, viewed as a map $\mathrm{BSTOP}(m) \xrightarrow{e} K(\mathbb{Z}, m)$ into an Eilenberg-MacLane space. The Euler class yields thus a homomorphism

$$\pi_m(\mathrm{BSTOP}(m)) \xrightarrow{e_\#} \pi_m(K(\mathbb{Z}, m)) = H^m(\mathbb{S}^m) \xrightarrow[\cong]{\langle -,[\mathbb{S}^m]\rangle} \mathbb{Z}.$$

If $m \neq 4, 8$, then the image of this map is $2\mathbb{Z}$ by Adams' result (see Proposition 2.3). The Euler class of the tangent bundle of $\mathbb{S}^m$ is $2x$, so the exact sequence above splits: we have constructed a left inverse for the second arrow $\pi_m(\mathbb{S}^m) \longrightarrow \pi_m(\mathrm{BSTOP}(m))$.

For $m = 4, 8$, we consider the last two rows of the big diagram,

$$\begin{tikzcd}
0 \arrow[r] & \pi_m(\mathbb{S}^m) \arrow[r] \arrow[d, equal] & \pi_m(\mathrm{BTOP}(m)) \arrow[r] \arrow[d] & \pi_m(\mathrm{BTOP}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_m(\mathbb{S}^m) \arrow[r] & \pi_m(\mathrm{BG}(m)) \arrow[r] & \pi_m(\mathrm{BG}) \arrow[r] & 0.
\end{tikzcd}$$

Let T'_m denote the torsion subgroup in $\pi_m(\mathrm{BTOP}(m))$, and $T_m \cong \mathbb{Z}/(m/2)$ the torsion subgroup of $\pi_m(\mathrm{BTOP})$). It is clear that T'_m injects into T_m. The sequence splits if and only if T'_m maps isomorphically onto T_m. Now $\pi_4(\mathrm{BG}(4)) \cong \pi_3(\mathrm{G}(4)) \cong \pi_7(\mathbb{S}^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12$, and $\pi_4(\mathrm{BG}) \cong \pi_3^s(\mathbb{S}^0) \cong \mathbb{Z}/24$. Similarly, $\pi_8(\mathrm{BG}(8)) \cong \pi_7(\mathrm{G}(8)) \cong \pi_{15}(\mathbb{S}^8) \cong \mathbb{Z} \oplus \mathbb{Z}/120$, and $\pi_8(\mathrm{BG}) \cong \pi_7^s(\mathbb{S}^0) \cong \mathbb{Z}/240$. See Toda [60] Ch. XIV, Hu [27] Ch. XI. Theorem 16.4 and p. 332, or Fomenko-Fuchs-Gutenmacher [18] p. 300 for these groups. It follows that in the bottom row of the diagram, a generator $\iota_m \in \pi_m(\mathbb{S}^m)$ maps to $(2, -1) \in \mathbb{Z} \oplus \mathbb{Z}/12$

resp. in $\mathbb{Z} \oplus \mathbb{Z}/120$, for $m = 4, 8$. If we tensor the diagram above with $\mathbb{Z}/2$ for $m = 4$ (resp. with $\mathbb{Z}/4$ for $m = 8$), then the image of ι_m still has order 2 (resp. 4) in $\pi_m(\mathrm{BG}(m)) \otimes \mathbb{Z}/(m/2)$. Thus the bottom row remains exact after tensoring, and therefore, the upper row remains also exact (the only point to be checked was the injectivity of the second arrow). It follows that $T'_m \cong T_m$. q.e.d.

In dimension $m = 2$, we use Kneser's old result $\mathrm{BO}(2) \simeq \mathrm{BTOP}(2)$, see Kirby-Siebenmann [33] p. 254. Thus, there is a (non-split) short exact sequence

$$0 \longrightarrow \pi_2(\mathbb{S}^2) \longrightarrow \pi_2(\mathrm{BTOP}(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and $\pi_2(\mathrm{BTOP}(2)) \cong \mathbb{Z}$. Using similar ideas as above, it is not difficult to prove that the sequence

$$0 \longrightarrow \pi_m(\mathbb{S}^m) \longrightarrow \pi_m(\mathrm{BG}(m)) \longrightarrow \pi_m(\mathrm{BG}) \longrightarrow 0$$

splits for all even $m \neq 2, 4, 8$ (and for these three values, the sequence is not split). We will not need this result.

6.3. *Proof of Proposition* 3.3. Let ξ and ξ' be $\mathbb{R}^m$-bundles over $\mathbb{S}^m$, for $m \geq 2$ even. Assume first that we can choose orientations of these bundles such that $e(\xi) = e(\xi')$. Let c denote the classifying map for the oriented tangent bundle $\tau\mathbb{S}^m$, and let c_ξ and $c_{\xi'}$ be classifying maps for the oriented bundles ξ and ξ'. The splitting of the exact sequence

$$0 \longrightarrow \pi_m(\mathbb{S}^m) \xrightarrow{c_\#} \pi_m(\mathrm{BSTOP}(m)) \longrightarrow \pi_m(\mathrm{BSTOP}) \longrightarrow 0$$

implies then that $(c_\xi)_\# = (c_{\xi'})_\#$ (as maps $\pi_m(\mathbb{S}^m) \longrightarrow \pi_m(\mathrm{BSTOP})$). Thus $\xi \cong \xi'$.

In the general case $m \geq 4$, we have the action of $\pi_1(\mathrm{BO}(m)) \cong \pi_1(\mathrm{BTOP}(m)) \cong \mathbb{Z}/2$ on the higher homotopy groups. The generator α_0 of the fundamental group of $\mathrm{BO}(m)$ maps $c_\#$ to its negative $-c_\#$ (see Steenrod [57] 23.11). From the splitting of the exact sequence

$$0 \longrightarrow \pi_m(\mathbb{S}^m) \longrightarrow \pi_m(\mathrm{BTOP}(m)) \longrightarrow \pi_m(\mathrm{BTOP}) \longrightarrow 0$$

and the diagram in Proposition 6.1 we see that α_0 changes the sign of the Euler class. Thus, if $|e| = |e'|$, then we may as well assume that $e(\xi) = e(\xi')$.

The case $m = 2$ follows directly from $\mathrm{BO}(2) \simeq \mathrm{BTOP}(2)$. q.e.d.

We summarize our classification of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$, for $m = 2, 4, 8$, as follows.

Proposition 6.4. *Let ξ be an $\mathbb{R}^2$-bundle over $\mathbb{S}^2$. Up to equivalence, ξ is determined by its absolute Euler number $|e| = |\langle e(\xi), [\mathbb{S}^2]\rangle|$, and for each $|e| \in \mathbb{N}$, there exists one such bundle. A weak equivalence between any two such bundles is an equivalence.*

Let ξ be an $\mathbb{R}^4$-bundle over $\mathbb{S}^4$. Up to equivalence, ξ is determined by its absolute Euler number $|e| = |\langle e(\xi), [\mathbb{S}^2]\rangle|$, its Kirby-Siebenmann number $\langle \mathrm{ks}(\xi), [\mathbb{S}^4]\rangle \in \mathbb{Z}/2$ and the Pontrjagin number $\langle p_4(\xi), [\mathbb{S}^4]\rangle \in 2 \cdot \mathbb{Z}$. For each triple $(|e|, \mathrm{ks}, p_4) \in \mathbb{N} \times \mathbb{Z}/2 \times 2 \cdot \mathbb{Z}$ satisfying the relation $p_4 + 2|e| \equiv 0 \pmod 4$, there exists one such bundle. If two such bundles ξ, ξ' are weakly equivalent, but not equivalent, then $(|e|, p_4, \mathrm{ks}) = (|e'|, -p_4', \mathrm{ks}')$.

Let ξ be an $\mathbb{R}^8$-bundle over $\mathbb{S}^8$. Up to equivalence, ξ is determined by its absolute Euler number $|e| = |\langle e(\xi), [\mathbb{S}^2]\rangle|$, the number $\langle \kappa(\xi), [\mathbb{S}^8]\rangle \in \mathbb{Z}/4$ and the Pontrjagin number $\langle p_8(\xi), [\mathbb{S}^8]\rangle \in \frac{6}{7} \cdot \mathbb{Z}$. For each triple in $\mathbb{N} \times \mathbb{Z}/4 \times \frac{6}{7} \cdot \mathbb{Z}$ satisfying the relation $\frac{7}{3}p_8 + 2|e| \equiv 0 \pmod 4$, there exists one such bundle. If two such bundles ξ, ξ' are weakly equivalent, but not equivalent, then $(|e|, p_8, \kappa) = (|e'|, -p_8', -\kappa)$.

Proof. We prove the 8-dimensional case; the other cases are similar. First, we classify oriented bundles; the orientation we choose is the orientation determined by the universal oriented $\mathbb{R}^8$-bundle over BSTOP(8). Then it is clear from our discussion that ξ is determined by the data

$$(\langle e(\xi), [\mathbb{S}^8]\rangle, \langle \kappa(\xi), [\mathbb{S}^8]\rangle, \langle p_8(\xi), [\mathbb{S}^8]\rangle) \in \mathbb{Z} \times \mathbb{Z}/4 \times \tfrac{6}{7}\mathbb{Z}.$$

Now we have as in 6.3 the action of α_0 which changes the sign of the Euler class without changing the sign of p_8 and κ (since these two classes come from BTOP, where α_0 acts trivially). This shows that the given numbers classify the bundle up to equivalence.

Let $\iota_8 \in \pi_8(\mathbb{S}^8)$ denote the canonical generator, and let c be the classifying map for the oriented tangent bundle of $\mathbb{S}^8$. For the number-theoretic relation between the Pontrjagin class and the Euler class, we note first that the image $(c)_\#(\pi_8(\mathbb{S}^8))$ is a direct factor in $\pi_8(\mathrm{BSTOP}(8))$. The octonionic Hopf line bundle $\eta_{\mathbb{O}}$ has Euler class x and Pontrjagin class $6x$ (for a suitable generator x of $H^m(\mathbb{S}^m)$), see, e.g., [36] Theorem 9. Let h be a classifying map for the oriented bundle $\eta_{\mathbb{O}}$. Then $c_\#(\iota_8)$ and $h_\#(\iota_8)$ span $\pi_8(\mathrm{BSTOP}(8)) \otimes \mathbb{Q}$ (over $\mathbb{Q}$), and the image of h in $\pi_8(\mathrm{BSTOP})$ spans $\pi_8(\mathrm{BSTOP}) \otimes \mathbb{Q}$ (over $\mathbb{Q}$). Since $e(\xi)$ is necessarily integral for any oriented $\mathbb{R}^8$-bundle over $\mathbb{S}^8$, we see from Lemma 5.5 that there exists a bundle η' with classifying map h' and $e(\eta') = x$, $p_8(\eta') = \frac{6}{7}x$, and that $c_\#(\iota_8)$ and $h'_\#(\iota_8)$ span a direct complement of the torsion group of $\pi_8(\mathrm{BSTOP}(8))$ (over $\mathbb{Z}$).

Finally, a weak equivalence which is not an equivalence comes from a map $\mathbb{S}^8 \longrightarrow \mathbb{S}^8$ of degree -1; such a map changes the sign of every characteristic class. q.e.d.

7. The classification of the models

In this section, we obtain the final homeomorphism classification of our models. We fix some notation. Let ξ be an $\mathbb{R}^m$-bundle over $\mathbb{S}^m$, with absolute Euler number $|e| = 1$, for $m = 2, 4, 8$, let $M(\xi)$ be its Thom space, and let $s_0 : \mathbb{S}^m \longrightarrow E \subseteq M(\xi)$ be the zero-section. Let $y_m \in H^m(M(\xi))$ denote a generator, such that $x = s_0^\bullet y_m$ is a generator dual to the chosen orientation $[\mathbb{S}^m]$. For $m = 4, 8$, we have by 3.7 the relations

$$\begin{aligned} s_0^\bullet(p_m(M(\xi))) &= p_m(\xi) && (m = 4, 8) \\ s_0^\bullet(\mathrm{ks}(M(\xi))) &= \mathrm{ks}(\xi) && (m = 4) \\ s_0^\bullet(\kappa(M(\xi))) &= \kappa(\xi) && (m = 8). \end{aligned}$$

Theorem 7.1. *Up to homeomorphism, our construction yields precisely the following models.*

For $m = 2$, there is just one model, the complex projective plane $M(\eta_\mathbb{C}) \cong \mathbb{C}\mathrm{P}^2$, where $\eta_\mathbb{C}$ is the complex Hopf line bundle (the tautological bundle) over $\mathbb{C}\mathrm{P}^1 = \mathbb{S}^2$.

For $m = 4$, let $p_4(\xi) = 2(1+2t)x$ and $\mathrm{ks}(\xi) = sx$, for $(t, s) \in \mathbb{Z} \times \mathbb{Z}/2$. By Proposition 6.4, the pair $(r, s) = (1 + 2t, s)$ determines ξ up to equivalence, so we may put $M_{r,s} = M(\xi)$. If there is a homeomorphism $M_{r,s} \cong M_{r',s'}$, then $(r, s) = (\pm r', s)$. The quaternion projective plane is $M_{1,0} \cong \mathbb{H}\mathrm{P}^2 = M(\eta_\mathbb{H}) \cong M_{-1,0}$. The model $M_{r,s}$ admits a PL *structure (unique up to isotopy) if and only if $s = 0$.*

For $m = 8$, let $p_8(\xi) = \frac{6}{7}(1+2t)x$ and $\kappa(\xi) = sx$, for $(t, s) \in \mathbb{Z}\times\mathbb{Z}/4$. Again, the pair $(r, s) = (1 + 2t, s)$ determines ξ up to equivalence by Proposition 6.4, so we may put $M_{r,s} = M(\xi)$. If there is a homeomorphism $M_{r,s} \cong M_{r',s'}$, then $(r, s) = \pm(r', s')$. The octonionic projective plane is $M_{7,0} \cong \mathbb{O}\mathrm{P}^2 = M(\eta_\mathbb{O}) \cong M_{-7,0}$. Each model $M_{r,s}$ admits a PL *structure (unique up to isotopy).*

Proof. The topological result is clear from our classification of $\mathbb{R}^m$-bundles over $\mathbb{S}^m$ with absolute Euler number $|e| = 1$. For a closed manifold M of dimension at least 5, the only obstruction to the existence of a PL structure is the Kirby-Siebenmann class $\mathrm{ks}(M)$. If such a PL structure exists on M , the number of isotopy classes of PL structures is determined by $[M; \mathrm{TOP}/\mathrm{PL}] \cong H^3(M; \mathbb{Z}/2)$, see Kirby-Siebenmann [33] p. 318. In our case, these groups are zero. q.e.d.

The models constructed by Eells-Kuiper in [14] are $\mathbb{C}\mathrm{P}^2$ (for $m = 2$), the models $M_{r,0}$, with $r = 1 + 2t$ (for $m = 4$), and the models $M_{7r,0}$, with $r = 1 + 2t$ (for $8 = 4$). Brehm-Kühnel [6] construct 8-dimensional PL manifolds which look like projective planes, and with small numbers of vertices. However, the characteristic classes of their examples seem to be unknown.

7.2. Now we consider the question which of the models admit a DIFF structure. The number of PL or DIFF structures on $\mathbb{C}\mathrm{P}^2$ is presently not known, so we concentrate from now on the cases $m = 4, 8$. Concerning DIFF structures in higher dimensions, a necessary condition (besides the existence of a PL structure) is clearly that $\tau M(\xi)$ admits a vector bundle structure. In particular, $\xi \cong s_0^* \tau M(\xi)$ has to admit a vector bundle structure. Thus, if $M_{r,s} = M_{1+2t,s}$ admits a DIFF structure, then clearly $s = 0$, and in addition $t \equiv 3 \pmod 7$ for $m = 8$. But this guarantees only that ξ admits a vector bundle structure. From 4.6 and Lemma 4.7, we see that

$$\widehat{\mathcal{A}}\,[M_{1+2t,s}] = -\frac{t(1+t)}{2^3 \cdot 7} \qquad (m = 4)$$

$$\widehat{\mathcal{A}}\,[M_{7(1+2u),s}] = -\frac{u(1+u)}{2^7 \cdot 127} \qquad (m = 8).$$

Since $M_{r,s}$ is 3-connected, the first Stiefel-Whitney classes vanish. Thus, if $M_{r,s}$ admits a DIFF structure, then it is a Spin manifold, see Lawson-Michelsohn [39] Ch. II Theorem 2.1. But for a closed oriented Spin manifold M^{4k}, the $\widehat{\mathcal{A}}$-genus $\widehat{\mathcal{A}}[M]$ is precisely the index of the Atiyah-Singer operator, see Lawson-Michelsohn [39] Ch. IV Theorem 1.1; in particular, it is an integer.

Lemma 7.3. *If $M_{r,s}$ admits a* DIFF *structure, then we have the following relations: For $m = 4$ put $r = 1 + 2t$. Then $s = 0$ and $t \equiv 0, 7, 48, 55 \pmod{56}$. For $m = 8$ put $r = 7(1 + 2u)$. Then $s = 0$ and u is an integer with $u \equiv 0, 127, 16128, 16255 \pmod{16256}$.*

This corresponds to Eells-Kuiper [14] Proposition 10 A, p. 43. In fact, the above conditions are sharp.

Theorem 7.4. *If $m = 4$, then $M_{1+2t,s}$ admits a* DIFF *structure if and only if $s = 0$ and $t \equiv 0, 7, 48, 55 \pmod{56}$. If $m = 8$, then $M_{7(1+2u),s}$ admits a* DIFF *structure if and only if $s = 0$ and $u \equiv 0, 127, 16128, 16255 \pmod{16256}$.*

Proof. The number theoretic condition guarantees that ξ admits a vector bundle structure. Thus we may choose a Riemannian metric on ξ. Let $SE \longrightarrow \mathbb{S}^m$ denote the corresponding unit sphere bundle of ξ. Then $E_0 \simeq SE$ is a homotopy $2m-1$-sphere and thus homeomorphic to $\mathbb{S}^{2m-1}$ by the proof of the generalized Poincaré conjecture, see Smale [54] and Newman [47]. If SE is diffeomorphic to $\mathbb{S}^{2m-1}$, then we may choose a diffeomorphism $\alpha : \mathbb{S}^{2m-1} \longrightarrow SE$. Gluing the closed $2m$-disk $\mathbb{D}^{2m}$ along α to the closed unit disk bundle DE of the vector bundle ξ, we obtain a smooth $2m$-manifold $DE \cup_\alpha \mathbb{D}^{2m}$ homeomorphic to $M_{r,s} \cong DE/SE$. Thus the problem is reduced to the question whether SE is diffeomorphic to $\mathbb{S}^{2m}$. Now the unit disk bundle $X = DE$ is an *almost closed manifold*, i.e., a smooth compact manifold X with boundary $\partial X = SE$ a homotopy sphere, see Wall [62]. By *loc.cit.* p. 178, such a manifold X^{2m} has a standard sphere $\mathbb{S}^{2m-1}$ as its boundary if and only if its $\widehat{\mathcal{A}}$-genus is integral, provided that $m = 4, 8$ and that X is $m-1$-connected. The $\widehat{\mathcal{A}}$-genus of DE coincides of course with the $\widehat{\mathcal{A}}$-genus of $M_{r,s}$. q.e.d.

The case $m = 8$ remained open in Eells-Kuiper [14]. Note that the existence of a positive scalar curvature metric implies that the $\widehat{\mathcal{A}}$-genus vanishes, see Lawson-Michelsohn [39] Ch. IV Theorem 4.1; this happens only for the models $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$.

Proposition 7.5. *The only models which admit a* DIFF *structure with a positive scalar curvature metric are $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$.*

These two manifolds admit a positive scalar curvature metric for *any* DIFF structure; this follows from Stolz' proof of the Gromov-Lawson-Rosenberg conjecture, see Stolz [59] Theorem A (for these two manifolds, Rosenberg's earlier result [51] for lower dimensions actually suffices). For spin manifolds of dimension $4k$, the map $\alpha : \Omega_{4k}^{\mathrm{Spin}} \longrightarrow \mathrm{KO}(\mathbb{S}^{4k})$ can be identified with a scalar multiple of the $\widehat{\mathcal{A}}$-genus.

Concerning the number of DIFF structures on a smoothable model $M_{r,s}$, we have Wall's result [62] which says that the DIFF structure on the almost closed manifold DE is unique. The group Θ_{2m} then acts transitively on the collection of all smoothings of $M_{r,s}$, see the introduction in Stolz [58]. This group is cyclic of order 2 for $m = 4, 8$, see Kervaire-Milnor [32] p. 504.

Proposition 7.6. *If a model $M_{r,s}$, for $m = 4, 8$, admits a* DIFF *structure, then it admits at most two distinct* DIFF *structures.*

This fact was already observed in Eells-Kuiper [14]. To obtain a more precise result, i.e., the exact number of DIFF structures, one would have to determine the inertia groups of the smoothable models (see Stolz [58]). Maybe his high-dimensional techniques can be adapted to this situation.

7.7. Finally, we consider oriented bordisms between distinct models. So suppose that

$$\partial W = M_{r,s} \cup -M_{r',s'}$$

is a compact oriented (topological) bordism between two models. Clearly, the numbers $p_m^2[M_{r,s}]$, $\mathrm{ks}^2[M_{r,s}]$ (for $m = 4$), and $\kappa^2[M_{r,s}]$ (for $m = 8$) are bordism invariants. Thus, the existence of an oriented bordism implies that $(r^2, s^2) = ({r'}^2, {s'}^2)$. This is good enough to settle the case $m = 4$; here, we conclude that $(r, s) = \pm(r', s')$ and thus $M_{r,s} \cong M_{r',s'}$, because $s \in \mathbb{Z}/2$ has no sign. For $m = 8$ this is not good enough to conclude that $(r, s) = \pm(r', s')$ because s is $\mathbb{Z}/4$-valued. So we use the standard fact from topological surgery theory (as developed in Kirby-Siebenmann [33]) that such a bordism W can be made 7-connected. The classifying map $W \longrightarrow \mathrm{BSTOP}$ for the oriented tangent bundle lifts thus to the 7-connected cover $\mathrm{BSTOP}\langle 8\rangle$. Put $\pi = \pi_8(\mathrm{BSTOP}) \cong \mathbb{Z} \oplus \mathbb{Z}/4$ and let $\mathrm{BSTOP}\langle 8\rangle \longrightarrow K(\pi, 8)$ denote the corresponding characteristic map. Let $x \in H^8(K(\pi, 8)) \cong \mathbb{Z} \oplus \mathbb{Z}/4$ be a generator for a free cyclic factor, and let q_8 denote its image in $H^8(\mathrm{BSTOP}\langle 8\rangle)$. If X is any 7-connected CW-complex, and if ξ is a stable $\mathbb{R}^n$-bundle over X, then the classifying map $X \xrightarrow{c} \mathrm{BSTOP}$ lifts

$$\begin{tikzcd} & \mathrm{BSTOP}\langle 8\rangle \arrow[r, "q_8"] \arrow[d] & K(\pi, 8) \\ X \arrow[ur, dotted] \arrow[r, "c"] & \mathrm{BSTOP} & \end{tikzcd}$$

and the class $q_8(\xi)$ is defined. From the coefficient pairing $\mathbb{Z} \otimes \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4$, we obtain for any $\mathbb{R}^n$-bundle over a 7-connected space X a $\mathbb{Z}/4$-valued 16-dimensional characteristic class $q_8\kappa$ and clearly, this class is a bordism invariant for 7-connected oriented bordisms. For $X = \mathbb{S}^8$, we know that $q_8(\xi) = \frac{7}{6}p_8(\xi)$. For our 16-dimensional models, we have thus $q_8(M_{r,s}) = \frac{7}{6}p_8(M)$ by 3.7, and this is an odd integral multiple of the generator $y_8 \in H^8(M_{r,s})$. The 7-connected bordism yields now the additional relation $\frac{7}{6}p_8\kappa[M_{r,s}] = \frac{7}{6}p_8\kappa[M_{r',s'}]$, which implies that $(r,s) = \pm(r',s')$.

Proposition 7.8. *Non-homeomorphic models $M_{r,s}$, $M_{r',s'}$ fall into different oriented bordism classes in $\Omega_{2m}^{\mathrm{STOP}}$.*

8. The homotopy types of the models

For the homotopy classification of our models we use the *Spivak fibration.* We recall the construction and refer to Klein [34] for more details. Let M be a closed 1-connected manifold (the case where $\pi_1(M) \neq 1$ is more involved and will not be important to us). There exists an embedding $M \hookrightarrow \mathbb{S}^N$, for some sufficiently large N, such that M hat a normal bundle νM in $\mathbb{S}^N$. In the group $\widetilde{\mathrm{KTOP}}(M) = [M; \mathrm{BTOP}]$, the bundle νM is just the inverse of the stable bundle class determined by the tangent bundle τM, since $\tau M \oplus \nu M \cong \tau\mathbb{S}^N|_M = \underline{\mathbb{R}}^N$. This shows that the normal bundle νM is unique up to stable equivalence.

There is a natural map $\mathbb{S}^N \xrightarrow{\alpha} M(\nu M)$, where $M(\nu M)$ is the Thom space of the normal bundle. Let $u(\nu M)$ be an orientation class. One shows that for the fundamental classes of $\mathbb{S}^N$ and M, one has the relation

$$u(\nu M) \frown \alpha_\bullet[\mathbb{S}^N] = [M]$$

(for the right choice of $u(\nu M)$). Now a result by Spivak [56] shows that the stable fiber homotopy type of the underlying spherical fibration σM of the bundle νM depends only on the homotopy type of M.

8.1. Let $c : M \longrightarrow \mathrm{BTOP}$ be a stable classifying map for νM, and let $d : M \longrightarrow \mathrm{BTOP}$ be a stable classifying map for τM. Then c is an inverse of d in the abelian group $\widetilde{\mathrm{KTOP}}(M) = [M; \mathrm{BTOP}]$, and so the composites $M \xrightarrow{c} \mathrm{BTOP} \longrightarrow \mathrm{BG}$ and $M \xrightarrow{d} \mathrm{BTOP} \longrightarrow \mathrm{BG}$ are inverse to each other in the abelian group $[M; \mathrm{BG}]$. But $M \xrightarrow{c} \mathrm{BTOP} \longrightarrow \mathrm{BG}$ is a classifying map for σM and depends thus by Spivak's result only on the homotopy type of M. It follows that the composite $M \xrightarrow{d} \mathrm{BTOP} \longrightarrow \mathrm{BG}$ is also a homotopy invariant of M.

Proposition 8.2. *If there is a homotopy equivalence $f : M(\xi) \xrightarrow{\simeq} M(\xi')$ between two models, then there is a fiber homotopy equivalence $\tau M(\xi) \oplus \underline{\mathbb{R}}^k \simeq f^*\tau M(\xi') \oplus \underline{\mathbb{R}}^k$, for some $k \geq 0$.*

8.3. Since $s_0 : \mathbb{S}^m \longrightarrow M(\xi)$ represents a generator of $\pi_m(M(\xi))$, this has the consequence that there is a fiber homotopy equivalence

$$\xi \oplus \underline{\mathbb{R}}^k \simeq g^*\xi' \oplus \underline{\mathbb{R}}^k,$$

for some homeomorphism $g : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ of degree ± 1. Since ξ has absolute Euler number $|e| = 1$, the mth Stiefel-Whitney class of ξ is nontrivial: the mth Stiefel-Whitney class is the mod 2 reduction of the Euler class, see Milnor-Stasheff [46] Proposition 9.5, so $0 \neq w_m(\xi) = x \in H^m(\mathbb{S};\mathbb{Z}/2)$. Also, the Stiefel-Whitney class depends only on the stable type of the spherical fibration of ξ. Let $R_m \subseteq \pi_m(\mathrm{BG})$ denote the set of all elements which represent a spherical fibration over $\mathbb{S}^m$ with nontrivial mth Stiefel-Whitney class. This is a coset of a subgroup of index 2 in $\pi_m(\mathrm{BG})$ (namely the kernel of the map $\pi_m(\mathrm{BG}) \xrightarrow{(w_m)_\#} \pi_m(K(m,\mathbb{Z}/2))$). Precomposing the classifying map with a map of degree -1, we achieve a change of sign for all elements in $\pi_m(\mathrm{BG})$. The group $\pi_m(\mathrm{BG}) \cong \pi^s_{m-1}(\mathbb{S}^0)$ is cyclic of order 2, 24, 240, for $m = 2, 4, 8$, respectively, see Toda [60] Ch. XIV, Hu [27] Ch. XI. Theorem 16.4 and p. 332, or Fomenko-Fuchs-Gutenmacher [18] p. 300. Thus we see that there are at least $1, 6, 60$ distinct homotopy types which are realized by our models, for $m = 2, 4, 8$. In the appendix we prove that these numbers are the precise numbers of homotopy types of Poincaré duality complexes (see 9.3) which look like projective planes.

Theorem 8.4. *Every 1-connected Poincaré duality complex which looks like a projective plane is homotopy equivalent to one of our models. The homotopy type of a model $M_{r,s}$ can be determined as follows. For $m = 2$, there is just one model and one homotopy type, namely $\mathbb{C}\mathrm{P}^2$.*

If $m = 4$, then $M_{r,s} \simeq M_{r',s'}$ if and only if $r + 12s \equiv \pm(r' + 12s')$ (mod 24).

If $m = 8$, then $M_{r,s} \simeq M_{r',s'}$ if and only if $r + 60s \equiv \pm(r' + 60s')$ (mod 240).

9. Our set of models is complete

In this section we prove that every manifold which looks like a projective plane is homeomorphic to one of our model manifolds $M(\xi)$ —

except for dimension 4, where one has precisely two such manifolds, the model manifold $\mathbb{C}\mathrm{P}^2 = M(\eta_\mathbb{C})$ and in addition the Chern manifold Ch^4 which is not a Thom space, see Theorem 9.1. This is covered by Freedman's classification [19] of closed 1-connected 4-manifolds.

Theorem 9.1. *There are precisely two 1-connected closed 4-manifolds M with $H_\bullet(M) \cong \mathbb{Z}^3$, the complex projective plane $\mathbb{C}\mathrm{P}^2$ and the Chern manifold Ch^4.*

Proof. This is stated in Freedman [19] pp. 370–372. Such a manifold is represented by the odd integral symmetric bilinear form $\omega = (1)$ on $H_2(M) \cong \mathbb{Z}$ and its Kirby-Siebenmann number $\mathrm{ks}[M] \in \mathbb{Z}/2$. q.e.d.

From now on, we assume that $m \neq 2$. The main result of this section is the following.

Theorem 9.2. *Let M^{2m} be a manifold which is like a projective plane. If $m \neq 2$, then M is homeomorphic to one of our models $M(\xi)$.*

The proof requires surgery techniques, so we recall the relevant notions. More information can be found in Madsen-Milgram [41] Ch. 2, in Kirby-Siebenmann [33] Essay V App. B, in Wall [64] Ch. 10, and in particular in Kreck [38]. The basic fact to keep in mind is that by the results of Kirby-Siebenmann [33], higher dimensional surgery works well in the topological category. The case $m = 2$ can in principle be handled by similar methods, see Freedman-Quinn [20].

9.3. The spaces we are dealing with are 1-connected, and this simplifies some points. Suppose that X is a finite and 1-connected CW-complex. Assume moreover that there is an element $[X] \in H_n(X)$ such that the cap product induces an isomorphism

$$H^k(X) \xrightarrow[\cong]{\frown [X]} H_{n-k}(X)$$

for all k. Then X satisfies Poincaré duality, and the pair $(X, [X])$ is what is called a *Poincaré duality complex* (of formal dimension n). Every closed, 1-connected and oriented manifold is a Poincaré duality complex (we consider here only the 1-connected case; the presence of a fundamental group requires the more complicated notion of a simple homotopy type).

9.4. Next, recall that an *h-cobordism* $(W; M_1, M_2)$ is a simply connected compact bordism between (simply connected) closed manifolds M_1, M_2,

$$\partial W = M_1 \dot{\cup} M_2,$$

with the property that the inclusions $M_1 \longrightarrow W \longleftarrow M_2$ are homotopy equivalences; an example is the product cobordism, $(M \times [0,1], M \times 0, M \times 1)$. In higher dimensions, this is in fact the only example.

9.5. ***h*-cobordism Theorem.** *Every h-cobordism $(W; M_1, M_2)$ with $\dim(W) \geq 5$ is a product bordism, $W \cong M_1 \times [0,1]$. In particular, there is a homeomorphism $M_1 \cong M_2$.*

Proof. For $\dim W \geq 6$, this is proved in Kirby-Siebenmann [33], but unfortunately not stated explicitly as a Theorem; see *loc. cit.* p. 113 and p. 320. For $\dim(W) \geq 7$, a proof is given by Okabe [48]. The case $\dim(W) = 5$ is proved in Freedman-Quinn [20], with some remarks on the higher dimensional case. q.e.d.

9.6. Suppose now that M is a closed oriented manifold of dimension at least 5, and that $f : M \longrightarrow X$ is a homotopy equivalence, with $f_\bullet[M] = [X]$. Then f is called a homotopy manifold structure on X; two such homotopy manifold structures $M_1 \xrightarrow{f_1} X \xleftarrow{f_2} M_2$ are called *equivalent* if there exists an h-cobordism $(W; M_1, M_2)$ and a map $F : W \longrightarrow X$ such that the diagram

$$\begin{tikzcd} M_1 \arrow[r, hook] \arrow[dr, "f_1"'] & W \arrow[d, "F"] & M_2 \arrow[l, hook'] \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

commutes. This relation is transitive and symmetric; the set of all equivalence classes of homotopy manifold structures on X is the *structure set* $\mathcal{S}_{\mathrm{TOP}}(X)$. Since we are assuming that $\dim(M) \geq 5$, the h-cobordism Theorem 9.5 applies, and thus every element of $\mathcal{S}_{\mathrm{TOP}}(X)$ represents a well-defined homeomorphism type of a closed manifold homotopy equivalent to X.

Let $\mathrm{Aut}(X) \subseteq [X; X]$ denote the group of all self-equivalences of X. If $\mathcal{S}_{\mathrm{TOP}}(X)$ is nonempty, there is a natural action of $\mathrm{Aut}(X)$ on $\mathcal{S}_{\mathrm{TOP}}(X)$, and the orbit set

$$\mathcal{M}_{\mathrm{TOP}}(X) = \mathcal{S}_{\mathrm{TOP}}(X)/\mathrm{Aut}(X)$$

can be identified with the set of all homeomorphism types of manifolds homotopy equivalent with X.

9.7. In order to determine the structure set $\mathcal{S}_{\mathrm{TOP}}(X)$, it is convenient to introduce yet another set which contains $\mathcal{S}_{\mathrm{TOP}}(X)$ as a subset, the set $\mathcal{T}_{\mathrm{TOP}}(X)$ of *tangential invariants*. Recall from Section 8 that associated to a 1-connected Poincaré duality complex X is a spherical fibration, the *Spivak normal bundle* σX whose stable fiber homotopy class depends only on the homotopy type of X. Let $S\tau M$ denote the spherical fibration of the topological tangent bundle τM. If $f : M \longrightarrow X$ is a homotopy equivalence, then the spherical fibration $f^*\sigma X \oplus S\tau M$ is stably fiber homotopically trivial. To put it differently, $f^*\sigma X$ is the stable inverse of the spherical fibration $S\tau(M)$ of the tangent bundle in the fiber homotopy category. Let $\sigma_T X$ be a stable inverse of σX. Then $f^*\sigma_T X$ is stably fiber homotopy equivalent to $S\tau(M)$; in particular, the stable fiber homotopy type of $S\tau(M)$ is a homotopy invariant of M. We used this fact already in Section 8. To make things more concrete, let us say that $\sigma_T X$ is an N-spherical fibration, for $N > 2\dim(M)$. Then $f : M \longrightarrow X$ induces a bundle map $\tau M \oplus \underline{\mathbb{R}}^{N-\dim(M)} \longrightarrow f^*\sigma_T X$ which is a fiber homotopy equivalence. Such a map is called a TOP *reduction* of $f^*\sigma_T X$. Two reductions are called equivalent if they differ by a fiber homotopy equivalence; the set of all stable TOP reductions of a spherical fibration ϕ is denoted $\mathrm{RTOP}(\phi)$. One can show that every element of $\mathcal{S}_{\mathrm{TOP}}(X)$ yields a well-defined reduction of $\sigma_T(X)$; this correspondence is injective, and we obtain an injection $\mathcal{S}_{\mathrm{TOP}}(X) \longrightarrow \mathrm{RTOP}(\sigma_T X)$. We call $\mathcal{T}_{\mathrm{TOP}}(X) = \mathrm{RTOP}(\sigma_T X)$ the set of *tangential invariants* of X. (Most texts consider normal invariants instead of tangential invariants. Since we are working in the stable category, the difference is merely the sign. Kirby-Siebenmann [33] use tangential invariants.)

Given a spherical fibration ϕ which admits a TOP reduction, it can be shown that the abelian group $[X; \mathrm{G/TOP}]$ acts regularly on $\mathrm{RTOP}(\phi)$; thus there is a bijection of sets

$$\mathcal{T}_{\mathrm{TOP}}(X) \cong [X; \mathrm{G/TOP}].$$

Now we can state the surgery classification of manifolds of a given homotopy type. So let X be a 1-connected finite Poincaré duality complex of formal dimension $n \geq 5$. There exists an abelian group P_n and a map $\theta : \mathcal{T}_{\mathrm{TOP}}(X) \longrightarrow P_n$, such that $\mathcal{S}_{\mathrm{TOP}}(X)$ is precisely the preimage

$\theta^{-1}(0)$; in other words, there is an exact sequence of sets

$$\begin{tikzcd} \mathcal{S}_{\mathrm{TOP}}(X) \arrow[r, hook, "\mathrm{inj}"] & \mathcal{T}_{\mathrm{TOP}}(X) \arrow[r, "\theta"] & P_n \\ & {[X;\mathrm{G/TOP}]} \arrow[u] \arrow[ur, dotted] & \end{tikzcd}$$

the *surgery exact sequence*. The dotted arrow is in general not a group homomorphism (and $\mathcal{T}_{\mathrm{TOP}}(X)$ has no canonical group structure). If $n = 4k \geq 8$ (the case we are interested in), then $P_n \cong \mathbb{Z}$ and θ is connected to the $\mathcal{L}$-genus and the signature as

$$\theta(\xi) = \frac{1}{8}(\langle \mathcal{L}_{4k}(\xi), [X]\rangle - \mathrm{Sig}(X)).$$

In other words, a bundle ξ represents a homotopy manifold structure for X if and only if ξ satisfies Hirzebruch's signature theorem.

Using these techniques, the proof of Theorem 9.2 is accomplished by the following steps.

Step 1. Every 1-connected Poincaré duality complex P (of formal dimension $2m \geq 5$) which has the same homology as a projective plane is homotopy equivalent to one of our models $M(\xi)$.

Step 2. Let ϕ be a stable spherical fibration over a finite, 1-connected CW-complex X with the property that $H_k(X) = 0$ for $k \not\equiv 0 \pmod 4$. Then the Pontrjagin character ph injects $\mathrm{RTOP}(\phi)$ into $H^\bullet(X;\mathbb{Q})$. In particular, $\mathrm{ph} : \mathcal{T}_{\mathrm{TOP}}(P) \longrightarrow H^\bullet(P;\mathbb{Q})$ is an injection.

Step 3. For $m = 4, 8$, we show that the elements of $\mathcal{S}_{\mathrm{TOP}}(P)$ are completely determined by their Pontrjagin classes p_m, and that all possibilities for the p_m are realized through our models $M(\xi)$.

Step 4. We determine $\mathrm{Aut}(P)$ and $\mathcal{M}_{\mathrm{TOP}}(P)$.

9.8. In the remainder of this section, we carry out Steps 1–4. Let P be a 1-connected Poincaré duality complex of formal dimension $2m$ which is like a projective plane, for $m = 4, 8$. So $H_k(P) \cong \mathbb{Z}$ for $m = 0, 1, 2$. We fix a map

$$s : \mathbb{S}^m \longrightarrow P$$

representing a generator of $\pi_m(P) \cong H_m(P) \cong \mathbb{Z}$. Note also that P has a preferred orientation $[P]$ — the class dual to y_m^2, where $y_m \in H^m(P) \cong \mathbb{Z}$ is any generator. Thus, any homotopy equivalence automatically preserves fundamental classes. By Wall [63] Proposition 4.1, we may assume that

$$P = \mathbb{S}^m \cup_\alpha e^{2m}$$

is a 2-cell complex, and that $m = 4, 8$, see 10.1.

Proof for Step 1. This is just Theorem 8.4. However, we can say a bit more: the results in Section 8 show that the homotopy type of P is uniquely determined by the stable weak type of the spherical fibration $s^*\sigma_T P$, i.e., by stable type of the pair of spherical fibrations $\{s^*\sigma_T P, i^*s^*\sigma_T P\}$, where $i : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ is any map of degree -1.

Proof for Step 2. Let ϕ be a stable spherical fibration over a connected CW-complex X, with stable classifying map $c : X \longrightarrow \mathrm{BG}$. A (stable) TOP-reduction of c is a lift C

$$\begin{tikzcd} & \mathrm{BTOP} \arrow[d, "f_{\mathrm{G}}^{\mathrm{TOP}}"] \\ X \arrow[ur, "C"] \arrow[r, "c"'] & \mathrm{BG}, \end{tikzcd}$$

with $f_{\mathrm{G}}^{\mathrm{TOP}} \circ C = c$. Two such lifts C_0, C_1 are equivalent if there exists a homotopy $C : X \times [0,1] \longrightarrow \mathrm{BTOP}$ with $c = f_{\mathrm{G}}^{\mathrm{TOP}} \circ C_t$ for all $t \in [0,1]$, i.e., if the homotopy is constant when projected to BG. The set of equivalence classes of lifts of c is denoted $\mathrm{RTOP}(\phi)$. If c is the constant map (and thus $\phi = \underline{0}$ is trivial), we obtain a bijection $\mathrm{RTOP}(\underline{0}) \cong [X; \mathrm{G/TOP}]$. For $\eta \in \mathrm{RTOP}(\phi)$ and $\zeta \in \mathrm{RTOP}(\psi)$ we have $\eta \oplus \zeta \in \mathrm{RTOP}(\phi \oplus \psi)$. This establishes a bijection

$$\mathrm{RTOP}(\underline{0}) \longrightarrow \mathrm{RTOP}(\phi)$$
$$\eta \longmapsto \eta \oplus \zeta$$

see Wall [64] Sec. 10, p. 113. Thus, we can identify $\mathrm{RTOP}(\phi)$ with $[X; \mathrm{G/TOP}]$, provided that $\mathrm{RTOP}(\phi) \neq \emptyset$. Note however that $\mathrm{RTOP}(\phi)$ has in general no natural group structure; rather, the abelian group $[X; \mathrm{G/TOP}]$ acts regularly on this set.

In general, different elements of $\mathrm{RTOP}(\phi)$ can be equivalent when viewed as stable bundles. From the homotopy viewpoint, this is due to the fact that two lifts C, C' can be homotopic without being fiber homotopic. We prove now that under certain conditions on X, the map $\mathrm{RTOP}(\phi) \longrightarrow [X, \mathrm{BTOP}]$ is injective.

Proposition 9.9. *Let X be a finite* 1*-connected CW-complex. Assume that $H_k(X) = 0$ for all $k \not\equiv 0 \pmod 4$. Then the natural map* $[X; \mathrm{G/TOP}] \longrightarrow [X; \mathrm{BTOP}]$ *is injective.*

Before we start with the proof, we note the following. By Theorem 4.2, we have isomorphisms $\widetilde{\mathrm{KO}}(\mathbb{S}^k) \otimes \mathbb{Q} \longrightarrow \widetilde{\mathrm{KTOP}}(\mathbb{S}^k) \otimes \mathbb{Q}$ for all $k \geq 0$. By a well-known comparison theorem for half-exact cofunctors, see Dold [12] Ch. 7, or by the Atiyah-Hirzebruch spectral sequence, see Hilton [22] Ch. 3, this implies that

$$\widetilde{\mathrm{KO}}(Y) \otimes \mathbb{Q} \xrightarrow{\cong} \widetilde{\mathrm{KTOP}}(Y) \otimes \mathbb{Q}$$

is a natural isomorphism of homotopy functors for every finite connected CW-complex Y. We combine this with the Pontrjagin character ph, which is (by the same comparison theorem for half-exact cofunctors and by Bott Periodicity, see Section 4) rationally an isomorphism

$$\widetilde{\mathrm{KO}}(Y) \otimes \mathbb{Q} \xrightarrow[\cong]{\mathrm{ph}} \widetilde{H}^{4\bullet}(Y; \mathbb{Q})$$

to obtain an isomorphism

$$\widetilde{\mathrm{KTOP}}(Y) \otimes \mathbb{Q} \xrightarrow[\cong]{\mathrm{ph}} \widetilde{H}^{4\bullet}(Y; \mathbb{Q})$$

which we also denote by ph.

Proof of Proposition 9.9. It clearly suffices to show that the natural map

$$[X; \mathrm{G/TOP}] \longrightarrow \widetilde{\mathrm{KTOP}}(X) \otimes \mathbb{Q}$$

is an injection. First we note that this is true in general for $X = \mathbb{S}^{4k+t}$, for $t = -1, 0, 1$ (note that $\pi_{4k\pm 1}(\mathrm{G/TOP}) = 0$, by Theorem 4.1). Both $[-; \mathrm{G/TOP}]$ and $\widetilde{\mathrm{KTOP}}(-) \otimes \mathbb{Q}$ are half-exact cofunctors, so injectivity holds also for a wedge of spheres $X = \bigvee_1^r \mathbb{S}^{4k+t}$ (this is just the additivity of half-exact cofunctors).

For a general complex X as in the claim of Proposition 9.9, we proceed by induction. By standard obstruction theory, we may assume that X is complex whose cells all have dimensions divisible by 4, i.e., that $X^{(4k)} = X^{(4k+1)} = X^{(4k+2)} = X^{(4k+3)}$ for all k, see Wall [63] Proposition 4.1. So suppose that $X = X^{(4k)}$, and that $A = X^{(4k-1)} = X^{(4k-4)}$. The long exact sequence of the pair (X, A) shows that A

also satisfies the hypothesis of Proposition 9.9, and by induction, we may assume that the conclusion of the proposition holds for A. Now consider the Puppe sequence

$$\bigvee_r \mathbb{S}^{4k-1} \longrightarrow A \longrightarrow X \longrightarrow \bigvee_r \mathbb{S}^{4k} \longrightarrow SA \longrightarrow \cdots$$

Note that $\widetilde{\mathrm{KTOP}}(SA) \otimes \mathbb{Q} \cong \widetilde{H}^{4\bullet}(SA;\mathbb{Q}) = 0$, and that similarly $\widetilde{\mathrm{KTOP}}(\bigvee_r \mathbb{S}^{4k-1}) \otimes \mathbb{Q} = 0$. We thus obtain a diagram

$$\begin{tikzcd}
0 \arrow[d] & {[A;\mathrm{G/TOP}]} \arrow[l] \arrow[d, "\mathrm{inj}"'] & {[X;\mathrm{G/TOP}]} \arrow[l] \arrow[d] & {[\bigvee_r \mathbb{S}^{4k};\mathrm{G/TOP}]} \arrow[l] \arrow[d, "\mathrm{inj}"'] & {[SA;\mathrm{G/TOP}]} \arrow[l] \arrow[d, "\mathrm{inj}"'] \\
0 & \widetilde{\mathrm{KTOP}}(A) \otimes \mathbb{Q} \arrow[l] & \widetilde{\mathrm{KTOP}}(X) \otimes \mathbb{Q} \arrow[l] & \widetilde{\mathrm{KTOP}}(\bigvee_r \mathbb{S}^{4k}) \otimes \mathbb{Q} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

and this implies by the Five-Lemma that $[X;\mathrm{G/TOP}] \longrightarrow \widetilde{\mathrm{KTOP}}(X) \otimes \mathbb{Q}$ is injective, see, e.g., Eilenberg-Steenrod [16] Lemma 4.4. This finishes the proof of Proposition 9.9. q.e.d.

Corollary 9.10. *Let ϕ be a spherical fibration over a finite 1-connected CW-complex X, with $H_k(X) = 0$ for all $k \not\equiv 0 \pmod 4$, and assume that $\mathrm{RTOP}(\phi) \neq \emptyset$. Then $\mathrm{RTOP}(\phi)$ injects into $\widetilde{\mathrm{KTOP}}(X) \otimes \mathbb{Q}$ and, via the Pontrjagin character, into $\widetilde{H}^{4\bullet}(X;\mathbb{Q})$.*

Proof. Let $\zeta \in \mathrm{RTOP}(\phi)$ be a stable bundle. By Proposition 9.9, the Pontrjagin character injects $\mathrm{RTOP}(\underline{0})$ into $\widetilde{H}^{4\bullet}(X;\mathbb{Q})$. The diagram

$$\begin{tikzcd}
\mathrm{RTOP}(\underline{0}) \arrow[r, "{[\eta \longmapsto \eta + \zeta]}", "\mathrm{bij}"'] \arrow[d, "\mathrm{inj}"', "\mathrm{ph}"] & \mathrm{RTOP}(\phi) \arrow[d, "\mathrm{ph}"] \\
\widetilde{H}^{4\bullet}(X;\mathbb{Q}) \arrow[r, "{[x \longmapsto x + \mathrm{ph}(\zeta)]}", "\mathrm{bij}"'] & \widetilde{H}^{4\bullet}(X;\mathbb{Q})
\end{tikzcd}$$

commutes, and the claim follows. q.e.d.

Corollary 9.11. *Let X be a finite 1-connected Poincaré duality complex, and assume that $H_k(X) = 0$ for all $k \not\equiv 0 \pmod 4$. Then the Pontrjagin character ph injects the set $\mathcal{T}_{\mathrm{TOP}}(X)$ of tangential invariants into $\widetilde{H}^{4\bullet}(X;\mathbb{Q})$.* q.e.d.

This corollary applies in particular to our Poincaré duality complex P of formal dimension $2m$, for $m = 4, 8$. Note also the following. If $\eta, \zeta \in \mathrm{RTOP}(\phi)$ are elements with the same total Pontrjagin class, $p(\eta) = p(\zeta)$, then clearly $\mathrm{ph}(\eta) = \mathrm{ph}(\zeta)$. Therefore, the total Pontrjagin class p induces also an injection of $\mathrm{RTOP}(\phi)$ into $\widetilde{H}^{4\bullet}(X; \mathbb{Q})$.

Proof for Step 3. We use the same symbols P, y_m, y_{2m}, s as in 9.8. Since $\mathrm{Sig}(P) = 1$, a stable TOP-bundle reduction $\zeta \in \mathcal{T}_{\mathrm{TOP}}(P)$ represents a homotopy manifold structure for P if and only if

$$\mathcal{L}_{2m}(\zeta) = y_{2m}^2.$$

Also, we see from the formula for the $\mathcal{L}$-genus 4.6 that there are rational numbers c_m, d_m (depending only on m) such that

$$p_{2m}(\eta) = c_m \mathcal{L}_{2m}(\eta) + d_m p_m(\eta)^2$$

for any stable bundle η over P.

Lemma 9.12. *The map $\mathcal{T}_{\mathrm{TOP}}(P) \longrightarrow H^m(P; \mathbb{Q})$, $\zeta \longmapsto p_m(\zeta)$ is an injection when restricted to $\mathcal{S}_{\mathrm{TOP}}(P) \subseteq \mathcal{T}_{\mathrm{TOP}}(P)$.*

Proof. Let η, ζ be elements in $\mathcal{T}_{\mathrm{TOP}}(X)$ representing homotopy manifold structures, so $\mathcal{L}_{2m}(\eta) = y_m^2 = \mathcal{L}_{2m}(\zeta)$. If $p_m(\eta) = p_m(\zeta)$, then $p_{2m}(\eta) = p_{2m}(\zeta)$ by the formula above. Thus $p(\eta) = p(\zeta)$, whence $\mathrm{ph}(\eta) = \mathrm{ph}(\zeta)$, and therefore $\eta = \zeta$ by Corollary 9.10. q.e.d.

Now we prove that our set of model manifolds realizes all elements in $\mathcal{S}_{\mathrm{TOP}}(P)$. Let $\phi = s^* \sigma_T P$.

Lemma 9.13. *Let $\zeta \in \mathrm{RTOP}(\phi)$. Then there exists an $\mathbb{R}^m$-bundle ξ over $\mathbb{S}^m$ with absolute Euler number $|e| = 1$ which is stably equivalent to ζ.*

Proof. We have $w_m(\zeta) = x \bmod 2$ (because $w_m(M_{r,s}) = y_m \bmod 2$ for any model). From the split exact sequence in Lemma 6.2 we see that we can find an $\mathbb{R}^m$-bundle ξ over $\mathbb{S}^m$ with $w_m(\xi) = x \bmod 2$, and hence with any odd absolute Euler number. q.e.d.

Corollary 9.14. *For $m = 4, 8$, all elements of $\mathcal{S}_{\mathrm{TOP}}(X)$ are realized as model manifolds $M(\xi)$.*

Proof. Given a stable bundle $\eta \in \mathcal{T}_{\mathrm{TOP}}(P)$ representing a homotopy manifold structure $M \longrightarrow X$ in $\mathcal{S}_{\mathrm{TOP}}(X)$, we can find by Lemma 9.13 an oriented $\mathbb{R}^m$-bundle ξ over $\mathbb{S}^m$ which is stably equivalent to $s^*\eta$, with absolute Euler number $|e| = 1$. Then the model manifold $M(\xi)$ is homotopy equivalent to P by the remark in Step 1, because the homotopy types of P and $M(\xi)$ are determined by $s^*\sigma_T P$ and $s_0^*\sigma_T M(\xi)$, respectively. Composing the homotopy equivalence $M(\xi) \simeq P$ with a self-homeomorphism of $M(\xi)$ induced by a homeomorphism of degree -1 of $\mathbb{S}^m$, if necessary, we obtain a homotopy commutative diagram

$$\begin{tikzcd} \mathbb{S}^m \arrow[rr, "s"] \arrow[dr, "s_0"'] & & P \arrow[dl, "f"', "\simeq"] \\ & M(\xi). & \end{tikzcd}$$

By Lemma 9.12, $M(\xi)$ is precisely the homotopy manifold structure on P represented by η, because $f^\bullet p_m(M) = p_m(\eta)$, so $M \cong M(\xi)$. q.e.d.

This finishes Step 3.

Proof for Step 4. We proved already in Proposition 3.4 that $M(\xi) \cong M(\xi')$ if and only if ξ and ξ' are weakly equivalent. The group $\mathrm{Aut}(M(\xi))$ is cyclic of order two by Lemma 10.5, and this finishes the classification for $m = 4, 8$.

10. Appendix: Homotopy classification

In this section, all maps and homotopies are assumed to preserve base points.

10.1. Suppose that X is a 1-connected Poincaré duality complex of formal dimension n, with $H_\bullet(X) \cong \mathbb{Z}^3$ (9.3). Let $\mu \in H_n(X)$ denote the fundamental class. We have $H_0(X) \cong \mathbb{Z} \cong H_n(X)$, so $H_m(X) \cong \mathbb{Z}$ for some number $1 < m < n$. By the universal coefficient theorem, see Spanier [55] Ch. 5.5 Theorem 4, we have $H^j(X) \cong H_j(X)$ for all j. From Poincaré duality, we see that $n = 2m$, and that the map

$$H^m(X) \otimes H^m(X) \longrightarrow \mathbb{Z}, \qquad u \otimes v \longmapsto \langle u \smile v, \mu \rangle$$

is a duality pairing. Thus m is even, and $H^\bullet(X) \cong \mathbb{Z}[y_m]/(y_m^3)$, for some generator $y_m \in H^m(X)$. By Wall [63] Proposition 4.1, the CW-complex X is homotopy equivalent to a 2-cell complex,

$$X \simeq X_\alpha = \mathbb{S}^m \cup_\alpha e^{2m},$$

for some attaching map $\alpha : \mathbb{S}^{2m-1} \longrightarrow \mathbb{S}^m$. By Adams-Atiyah [2] Theorem A, this implies that $m = 2, 4, 8$. Note also that X_α has a preferred orientation μ, the dual of y_m^2.

10.2. We wish to determine the number of homotopy types of such complexes X. The structure of the cohomology ring of X_α implies that α, viewed as an element of $\pi_{2m-1}(\mathbb{S}^m)$, has Hopf invariant $h(\alpha) = \pm 1$, see Adams-Atiyah [2] and Husemoller [30] Ch. 20.10. Note also that the homotopy type of X_α is not changed if α is replaced by a map homotopic to α, see Milnor [45] Lemma 3.6. Let $\mathbf{W}_m$ denote the set of all homotopy types of 1-connected CW-complexes as above. Let $\mathcal{H}_m^{\pm 1} \subseteq \pi_{2m-1}(\mathbb{S}^m)$ denote the set of all elements of Hopf invariant ± 1. We thus have a well-defined surjection

$$\pi_{2m-1}(\mathbb{S}^m) \supseteq \mathcal{H}_m^{\pm 1} \longrightarrow \mathbf{W}_m,$$

sending $\alpha \in \mathcal{H}_m^{\pm 1}$ to the homotopy type of X_α. By Adams-Atiyah [2] Theorem A, the set $\mathbf{W}_m$ is empty unless $m = 2, 4, 8$.

Next we note the following. If $c : \mathbb{S}^{2m-1} \longrightarrow \mathbb{S}^{2m-1}$ is an involution of degree -1, then there is a homeomorphism $X_\alpha \cong X_{\alpha \circ c}$. Since $\alpha \circ c$ represents $-\alpha \in \pi_{2m-1}(\mathbb{S}^m)$, we have a homotopy equivalence

$$X_\alpha \simeq X_{-\alpha}.$$

Each element in $\mathbf{W}_m$ is thus represented by a element $\alpha \in \pi_{2m-1}(\mathbb{S}^m)$ with Hopf invariant $h(\alpha) = 1$, i.e., $\mathcal{H}_m^{+}$ surjects onto $\mathbf{W}_m$.

For $m = 2$, we are done: $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, see Toda [60] p. 186, so $\mathcal{H}_2^{\pm}$ has exactly two elements, and $\mathbf{W}_2$ consists of precisely one homotopy type, the complex projective plane $\mathbb{C}\mathrm{P}^2$.

Lemma 10.3. *There is exactly one homotopy type in* $\mathbf{W}_2$.

10.4. It remains to consider the cases $m = 4, 8$. Similarly as above, if $g : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ is an involution of degree -1, then there is a homeomorphism $X_\alpha \cong X_{g \circ \alpha}$, so

$$X_\alpha \simeq X_{g_\#(\alpha)}.$$

The homotopy equivalences

$$X_\alpha \simeq X_{-\alpha} \simeq X_{g_\#(\alpha)} \simeq X_{-g_\#(\alpha)}$$

are in fact the only homotopy equivalences which occur between these 2-cell complexes. For suppose that $f : X_\alpha \xrightarrow{\simeq} X_\beta$ is a homotopy equivalence. We may assume that f is a cellular map, see Whitehead [66] Ch. II Theorem 4.5,

$$f : (X_\alpha, \mathbb{S}^m) \longrightarrow (X_\beta, \mathbb{S}^m),$$

and then f restricts to a map of degree ± 1 on the m-skeleton $\mathbb{S}^m$. Also, we have the Hurewicz isomorphism

$$H_{2m}(\mathbb{S}^{2m}) = H_{2m}(X_\alpha/\mathbb{S}^m) \cong H_{2m}(X_\alpha, \mathbb{S}^m) \cong \pi_{2m}(X_\alpha, \mathbb{S}^m) \cong \mathbb{Z};$$

a canonical generator $\hat{\alpha}$ of this group $\pi_{2m}(X_\alpha, \mathbb{S}^m)$ is given by the attaching map,

$$(e^{2m}, \mathbb{S}^{2m-1}) \xrightarrow{\hat{\alpha}} (X_\alpha, \mathbb{S}^m) \quad \text{with} \quad \partial\hat{\alpha} = \alpha.$$

From the homotopy exact sequence

$$\begin{array}{ccccccccc}
\longrightarrow & \pi_{2m}(\mathbb{S}^m) & \longrightarrow & \pi_{2m}(X_\alpha) & \longrightarrow & \pi_{2m}(X_\alpha, \mathbb{S}^m) & \xrightarrow{\partial} & \pi_{2m-1}(\mathbb{S}^m) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \pi_{2m}(\mathbb{S}^m) & \longrightarrow & \pi_{2m}(X_\beta) & \longrightarrow & \pi_{2m}(X_\beta, \mathbb{S}^m) & \xrightarrow{\partial} & \pi_{2m-1}(\mathbb{S}^m) & \longrightarrow
\end{array}$$

and the Five-Lemma, we see that $f_\#(\hat{\alpha}) = \pm\hat{\beta}$, and so $f_\#(\alpha) = \pm\beta$. We have seen above that we may replace X_β by $X_{-\beta}$; thus, we may assume that $f_\#(\alpha) = \beta$. If f restricts to a map of degree 1 on $\mathbb{S}^m$, then $f_\#(\alpha) = \alpha = \beta$.

Lemma 10.5. *The group of self-equivalences* $\mathrm{Aut}(X_\alpha)$ *is cyclic of order* 2*; it coincides with the group of graded ring automorphisms of the cohomology ring* $\mathbb{Z}[y_m]/(y_m^3)$.

10.6. So the remaining problem is to determine the relation between α and $g_\#(\alpha)$, where $g : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ is a map of degree -1. Towards this end, we consider the EHP-sequence of $\mathbb{S}^n$ for the values $n = m-1, m$,

$$\longrightarrow \pi_k(\mathbb{S}^n) \xrightarrow{E} \pi_{k+1}(\mathbb{S}^{n+1}) \xrightarrow{H} \pi_{k+1}(\mathbb{S}^{2n+1}) \xrightarrow{P} \pi_{k-1}(\mathbb{S}^n) \xrightarrow{E}$$

This sequence is exact for $k \leq 3n-2$, see Whitehead [66] Ch. XII Theorem 2.2. Here, E is the suspension and H is the generalized Hopf invariant. Let $\iota_j = \mathrm{id}_{\mathbb{S}^j}$ denote the canonical generator of $\pi_j(\mathbb{S}^j)$. For $\rho \in \pi_{2n+1}(\mathbb{S}^n)$, one has $H(\rho) = h(\rho) \cdot \iota_{2n+1}$; see Whitehead [66] for a comparison between the various definitions of Hopf invariants.

10.7. Consider the diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & \pi_{2m-1}(\mathbb{S}^{2m-1}) \arrow[r, "E^2", "\cong"'] & \pi_{2m+1}(\mathbb{S}^{2m+1}) \arrow[d, "P"] \arrow[dr, "\cdot 2"] & & \\
0 \arrow[r] & \pi_{2m-2}(\mathbb{S}^{m-1}) \arrow[r, "E"] \arrow[dr] & \pi_{2m-1}(\mathbb{S}^{m}) \arrow[r, "H"] \arrow[d, "E"] & \pi_{2m-1}(\mathbb{S}^{2m-1}) \arrow[r] & 0 \\
 & & \pi_{2m}(\mathbb{S}^{m+1}) \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}
$$

The middle row is split and short exact: $\pi_{2m-1}(\mathbb{S}^{2m-1})$ is infinite cyclic (whence the splitting) and the Hopf invariant H is by assumption onto. The map $P : \pi_{2m}(\mathbb{S}^{2m-1}) \longrightarrow \pi_{2m-2}(\mathbb{S}^{m-1})$ can be characterized by $PE^2(\rho) = [\iota_{m-1}, \iota_{m-1}] \circ \rho$, see Whitehead [66] Ch. XII Theorem 2.4. But $\mathbb{S}^{m-1}$ is an H-space for $m = 4, 8$, so $[\iota_{m-1}, \iota_{m-1}] = 0$, see Whitehead [66] Ch. X Corollary 7.8. From the EHP sequence, we see that $\pi_{2m-2}(\mathbb{S}^{m-1}) \xrightarrow{E} \pi_{2m-1}(\mathbb{S}^m)$ is an injection.

The middle column is also short exact: from Freudenthal's Suspension Theorem we have that $E : \pi_{2m-1}(\mathbb{S}^m) \longrightarrow \pi_{2m}(\mathbb{S}^{m+1})$ is an epimorphism, see Whitehead [66] Ch. VII Theorem 7.13. To see that $P : \pi_{2m+1}(\mathbb{S}^{2m+1}) \longrightarrow \pi_{2m-1}(\mathbb{S}^m)$ is injective, note that $PE^2(\iota_{2m-1}) = [\iota_m, \iota_m]$ by Whitehead [66] Ch. XII Theorem 2.4. But

$$H([\iota_m, \iota_m]) = 2\iota_{2m-1},$$

see Whitehead [66] Ch. XI Theorem 2.5. Thus, P is injective on this infinite cyclic group.

10.8. So suppose that $g : \mathbb{S}^m \longrightarrow \mathbb{S}^m$ has degree -1. Then

$$g_\#([\iota_m, \iota_m]) = [g_\#(\iota_m), g_\#(\iota_m)] = [-\iota_m, -\iota_m] = [\iota_m, \iota_m],$$

whence

$$H(g_\#(\xi)) = H(\xi) \text{ for all } \xi \in \pi_{2m-1}(\mathbb{S}^m).$$

Now let $\rho \in \pi_{2m-2}(\mathbb{S}^{m-1})$. This group is finite by Serre's finiteness result for odd spheres, see Spanier [55] Ch. 9.7 Theorem 7; via E, we can identify it with the torsion group of $\pi_{2m-1}(\mathbb{S}^m)$. The double suspension E^2 injects this group into the stable group $\pi_{2m}(\mathbb{S}^{m+1}) = \pi^s_{m-1}(\mathbb{S}^0)$. In the graded ring $\pi^s_\bullet(\mathbb{S}^0)$, composition is commutative, see Whitehead [66] Ch. XII; thus we have $E(g_\#(E\rho)) = -E^2(\rho)$, whence $g_\#(E(\rho)) = -E(\rho)$. The involution $g_\#$ thus changes the signs of the elements of the torsion group of $\pi_{2m-1}(\mathbb{S}^m)$.

The stable groups $\pi^s_{m-1}(\mathbb{S}^0)$ are cyclic of order 24 and 240, for $m = 4, 8$, see Toda [60] p. 186. The image of the double suspension E^2 is also cyclic and has index 2 in this group. Thus, if $\xi \in \pi_{2m-1}(\mathbb{S}^m)$ is an element with Hopf invariant 1, then the element $E(\xi)$ together with the cyclic group $E^2(\pi_{2m-2}(\mathbb{S}^{m-1}))$ generates $\pi_{2m+2}(\mathbb{S}^{m+1}) \cong \pi^s_{m-1}(\mathbb{S}^0)$. Therefore, we can find an element $\eta_m \in \pi_{2m-1}(\mathbb{S}^m)$ with Hopf invariant $h(\eta_m) = 1$ whose suspension $E(\eta_m)$ generates $\pi_{2m}(\mathbb{S}^{m+1})$. Let $\delta_m = 2\eta_m - [\iota_m, \iota_m]$; then $E(\delta_m) = 2 \cdot E(\eta_m)$ generates the cyclic group $E^2(\pi_{2m-2}(\mathbb{S}^{m-1}))$ (the suspension of the Whitehead product $[\iota_m, \iota_m]$ vanishes, see Whitehead [66] Ch. X Theorem 8.20), so δ_m generates the torsion group of $\pi_{2m-1}(\mathbb{S}^m)$. Put $g_\#(\eta_m) = \eta_m + r \cdot \delta_m$. Now

$$\begin{aligned} -\delta_m = g_\#(\delta_m) &= g_\#(2\eta_m - [\iota_m, \iota_m]) \\ &= 2\eta_m + 2r \cdot \delta_m - [\iota_m, \iota_m] = (1 + 2r) \cdot \delta_m, \end{aligned}$$

whence $2(1 + r)\delta_m = 0$. This leaves two possibilities for r, for $m = 4, 8$. But due to the commutativity of $\pi^s_\bullet(\mathbb{S}^0)$, we have also $E(g_\#(\eta_m)) = -E(\eta_m) = E(\eta_m) + r \cdot E(\delta_m) = (1 + 2r)E(\eta_m)$, which implies that $r = -1$, i.e., that

$$g_\#(\eta_m) = \eta_m - \delta_m.$$

We have proved the following result.

Proposition 10.9. *The number of homotopy types in* $\mathbf{W}_m$ *is* $1, 6$, 60*, for* $m = 2, 4, 8$.

This is a complete homotopy classification of manifolds and complexes which are like projective planes, and also the end of this paper.

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